

RATES OF MIXING FOR THE WEIL-PETERSSON GEODESIC FLOW II: EXPONENTIAL MIXING IN EXCEPTIONAL MODULI SPACES

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ABSTRACT. We establish exponential mixing for the geodesic flow $\varphi_t: T^1S \rightarrow T^1S$ of an incomplete, negatively curved surface S with cusp-like singularities of a prescribed order. As a consequence, we obtain that the Weil-Petersson flows for the moduli spaces $\mathcal{M}_{1,1}$ and $\mathcal{M}_{0,4}$ are exponentially mixing, in sharp contrast to the flows for $\mathcal{M}_{g,n}$ with $3g - 3 + n > 1$, which fail to be rapidly mixing. In the proof, we present a new method of analyzing invariant foliations for hyperbolic flows with singularities, based on changing the Riemannian metric on the phase space T^1S and rescaling the flow φ_t .

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INTRODUCTION

Let S be an oriented surface with finitely many punctures. Suppose that S is endowed with a negatively curved Riemannian metric and that in a neighborhood of each puncture the metric is “asymptotically modeled” on a surface of revolution obtained by rotating the curve $y = x^r$, for some $r > 2$, about the x -axis in \mathbb{R}^3 (where r may depend on the puncture). The results in this paper allow us to conclude that the geodesic flow on T^1S mixes exponentially fast.

Before stating the hypotheses precisely, we recall some facts about the metric on a surface R of revolution for the function $y = x^r$. This surface is negatively curved, incomplete and the curvature can be expressed as a function of the distance to the cusp point p_0 where $x = y = 0$. Denote by $\rho(\cdot, \cdot)$ the induced Riemannian path metric and $\delta: R \rightarrow \mathbb{R}_{\geq 0}$ the Riemannian distance to the cusp:

$$\delta(p) = \rho(p, p_0).$$

Then for $r > 1$, the Gaussian curvature on R has the following asymptotic expansion in δ , as $\delta \rightarrow 0$:

$$K(p) = -\frac{r(r-1)}{\delta(p)^2} + O(\delta(p)^{-1}).$$

Our main theorem applies to any incomplete, negatively curved surface with singularities of this form. More precisely, we have:

Theorem 1. *Let X be a closed surface, and let $\{p_1, \dots, p_k\} \subset X$. Suppose that the punctured surface $S = X \setminus \{p_1, \dots, p_k\}$ carries a C^5 , negatively curved Riemannian metric that extends to a complete distance metric ρ on X . Assume that the lift of this metric to the universal cover \tilde{S} is geodesically convex. Denote by $\delta_i: S \rightarrow \mathbb{R}_+$ the distance $\delta_i(p) = \rho(p, p_i)$, for $i = 1, \dots, k$.*

Assume that there exist $r_1, \dots, r_k > 2$ such that the Gaussian curvature K satisfies

$$K(p) = \sum_{i=1}^k -\frac{r_i(r_i-1)}{\delta_i(p)^2} + O(\delta_i(p)^{-1})$$

and

$$\|\nabla^j K(p)\| = \sum_{i=1}^k O(\delta_i(p)^{-2-j}),$$

for $j = 1, 2, 3$ and all $p \in S$.

Then the geodesic flow $\varphi_t: T^1S \rightarrow T^1S$ is exponentially mixing: there exist constants $c, C > 0$ such that for every pair of C^1 functions $u_1, u_2 \in L^\infty(T^1S, \text{vol})$, we have

$$\left| \int_{T^1S} u_1 u_2 \circ \varphi_t d\text{vol} - \int u_1 d\text{vol} \int u_2 d\text{vol} \right| \leq C e^{-ct} \|u_1\|_{C^1} \|u_2\|_{C^1},$$

for all $t > 0$, where vol denotes the Riemannian volume on T^1S (which is finite) normalized so that $\text{vol}(T^1S) = 1$.

The regularity hypotheses on u_1, u_2 are not optimal. See Corollary 5.2 in the last section for precise formulations.

Theorem 1 has a direct application to the dynamics of the Weil-Petersson flow, which is the geodesic flow for the Weil-Petersson metric $\langle \cdot, \cdot \rangle_{WP}$ of the moduli spaces $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g \geq 0$ and $n \geq 0$ punctures, defined for $3g - 3 + n \geq 1$. For a

discussion of the WP metric and properties of its flow, see the recent, related work [7]. As a corollary, we obtain the following result, which originally motivated this study.

Corollary 0.1. *The Weil-Petersson geodesic flow on $T^1\mathcal{M}_{(g,n)}$ mixes exponentially fast when $(g, n) = (1, 1)$ or $(0, 4)$.*

Proof of Corollary. Wolpert shows in [16] that the hypotheses of Theorem 1 are satisfied by the WP metric on $\mathcal{M}_{g,n}$, for $3g - 3 + n = 1$. \diamond

Mixing of the WP flow (for all (g, n)) had previously been established in [9]. For $(g, n) \notin \{(1, 1), (0, 4)\}$, the conclusions of Corollary 0.1 do *not* hold [7]: for every $k > 0$, there exist compactly supported, C^k test functions u_1, u_2 such that the correlation between u_1 and $u_2 \circ \varphi_T$ decays at best polynomially in T .

Remark: The geodesic convexity assumption in Theorem 1 can be replaced by a variety of other equivalent assumptions. For example, it is enough to assume that $\delta_i = \beta_i + o(\delta_i)$, where β_i is a convex function (as is the case in the WP metric). Alternatively, one may assume a more detailed expansion for the metric in the neighborhood of the cusps. For example, the assumptions near the cusp are satisfied for a surface of revolution for the function $y = u(x)x^r$, where $u: [0, 1] \rightarrow \mathbb{R}_{\geq 0}$ is C^5 , with $u(0) \neq 0$ and $r > 2$. One can easily formulate further perturbations of this metric outside the class of surfaces of revolutions for which the hypotheses of Theorem 1 hold near $\delta = 0$.

To simplify the exposition and reduce as much as possible the use of unspecified constants, we will assume in our proof that $k = 1$, so that S has only one cusp.

0.1. Discussion. In a landmark paper [10], Dolgopyat established that the geodesic flow for any negatively-curved compact surface is exponentially mixing. His techniques, building in part on earlier work of Ruelle, Pollicott and Chernov, have since been extracted and generalized in a series of papers, first by Baladi-Vallée [5], then Avila-Gouëzel-Yoccoz [3], and most recently in the work of Araújo-Melbourne [4], upon which this paper relies.

Ultimately, the obstructions to applying Dolgopyat's original argument in this context are purely technical, but to overcome these obstructions in any context is the heart of the matter. The solution to the analogous problem in the billiards context – exponential mixing for Sinai billiards of finite horizon – has only been recently established [6].

To prove exponential mixing using the symbolic-dynamical approach of Dolgopyat, Baladi-Vallée et. al., one constructs a section to the flow with certain analytic and symbolic dynamical properties. In sum, one seeks a surface $\Sigma \subset T^1S$ transverse to the flow φ_t in the three manifold T^1S on which the dynamics of the return map can be tightly organized.

In particular, we seek a return time function $R: \Sigma_0 \rightarrow \mathbb{R}_{>0}$ defined on a full measure subset $\Sigma_0 \subset \Sigma$, with $\varphi_{R(v)}(v) \in \Sigma$ for all $v \in \Sigma_0$ and so that the dynamics of $F: v \mapsto \varphi_{R(v)}(v)$ on Σ_0 are hyperbolic and can be modeled on a full shift on countably many symbols. For φ_t to be exponentially mixing, the function R must be constant along stable manifolds, have exponential tails and satisfy a non-integrability condition (UNI) (which will hold automatically if the flow φ_t preserves a contact form, as is the case here).

Whereas in [5] and [3] the map F is required to be piecewise uniformly C^2 , the regularity of F is relaxed to $C^{1+\alpha}$ in [4]. This relaxation in regularity might seem mild, but it is crucial in applications to nonuniformly hyperbolic flows with singularities.

The reason is that the surface Σ is required to be saturated by leaves of the (strong) stable foliation \mathcal{W}^s for the flow φ_t . The smoothness of the foliation \mathcal{W}^s thus dictates the smoothness of the surface Σ which then determines the smoothness of F (up to the smoothness of the original flow φ_t). *Even in the case of contact Anosov flows in dimension 3*, the foliation \mathcal{W}^s is no better than $C^{1+\alpha}$, for some $\alpha \in (0, 1)$, unless the flow is algebraic in nature.¹

While it has long been known that this $C^{1+\alpha}$ regularity condition holds for the stable and unstable foliations of contact Anosov flows in dimension 3, this is far from the case for singular and nonuniformly hyperbolic flows, even in low dimension. In the context of this paper, the geodesic flow φ_t is not even complete, and the standard singular hyperbolic theory fails to produce φ_t -invariant foliations \mathcal{W}^u and \mathcal{W}^s , let alone foliations with $C^{1+\alpha}$ regularity.

The flows φ_t considered here, while incomplete, bear several resemblances to Anosov flows. Most notably, there exist $D\varphi_t$ -invariant stable and unstable cone fields that are defined *everywhere* in T^1S . The angle between these cone fields tends to zero as the basepoint in T^1S approaches the singularity. The action of $D\varphi_t$ in these cones is strongly hyperbolic, with the strength of the hyperbolicity approaching infinity as the orbit comes close to the singularity.

The key observation in this paper is that by changing the Riemannian metric on T^1S and performing a natural time change in φ_t one obtains a volume-preserving *Anosov* flow on a *complete* Riemannian manifold of finite volume. This time change does not change orbits and has a predictable effect on stable and unstable bundles. One can apply all of the known machinery for Anosov flows to this rescaled flow, and transferring the information back to the original flow, one concludes that φ_t possesses invariant stable and unstable foliations \mathcal{W}^u and \mathcal{W}^s that are locally uniformly $C^{1+\alpha}$. This gives the crucial input in constructing the section Σ and return function R defined above.

In the setting of Weil-Petersson geometry, one can summarize the results of this time change: in the exceptional case $3g - 3 + n = 1$, the Weil-Petersson geodesic flow, when run at unit speed *in the Teichmüller metric* is (like the Teichmüller flow) an Anosov flow. For $3g - 3 + n > 1$, the WP flow is not Anosov, even when viewed in the Teichmüller metric (or an equivalent Riemannian metric such as in [15]), but it might be fruitful to study the flow from this perspective. We remark here that Hamenstädt [11] has recently constructed measurable orbit equivalences between the WP and Teichmüller geodesic flows for all $3g - 3 + n \geq 1$.

A different approach, using anisotropic function spaces, has been employed by Liverani to establish exponential mixing for contact Anosov flows in arbitrary dimension, even when the foliations \mathcal{W}^u and \mathcal{W}^s fail to be C^1 [14]. This method is more holistic (though no less technical) as the arguments take place in the manifold itself (not a section) and avoid symbolic dynamics. It would be interesting to attempt to import this machinery to the present context. This is the approach employed in the recent work of Baladi, Demers and Liverani on Sinai billiards in [6] mentioned above.

This paper is organized as follows. In Section 1 we recall some facts about geodesic flows and basic comparison lemmas for ODEs. In Section 2, we establish (under the hypotheses of Theorem 1) C^4 regularity for the functions δ_i measuring distance to the

¹This issue is bypassed in the application to the Teichmüller flow in [2, 3] because there the stable and unstable foliations are locally affine.

cusps in S . The arguments there bear much in common with standard proofs of regularity of Busemann functions in negative curvature, but additional attention to detail is required to obtain the correct order estimates on the size of the derivatives of the δ_i . In Section 3 we establish basic geometric properties of the surfaces considered here, in close analogy to properties of surfaces of revolution. These results refine some known properties of the Weil-Petersson metric.

Section 4 addresses the global properties of the flow φ_t . Here we construct a new Riemannian metric on T^1S , which we call the \star metric, in which T^1S is complete. Rescaling φ_t to be unit speed in the \star metric, we obtain a new flow ψ_t which we prove is Anosov, with *uniform* bounds on its first three derivatives (in the \star metric). We derive consequences of this, including ergodicity of φ_t and existence and $C^{1+\alpha}$ regularity of φ_t invariant unstable and stable foliations \mathcal{W}^s and \mathcal{W}^u .

In the final section (Section 5), we construct the section Σ to the flow and return time function R satisfying the hypotheses of the Araújo-Melbourne theorem. In essence this is equivalent to constructing a Young tower for the return map to Σ and is carried out using standard methods. Here the properties of geodesics established in Section 3 come into play in describing the dynamics of the return map of the flow to the compact part of T^1S .

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1. NOTATION AND PRELIMINARIES

Let S be an oriented surface endowed with a Riemannian metric. As usual $\langle v, w \rangle$ denotes the inner product of two vectors and ∇ is the Levi-Civita connection defined by the Riemannian metric. It is the unique connection that is symmetric and compatible with the metric.

The surface S carries a unique almost complex structure compatible with the metric. We denote this structure by J ; for $v \in T_p^1S$, the vector Jv is the unique tangent vector in T_p^1S such that (v, Jv) is a positively oriented orthonormal frame.

The covariant derivative along a curve $t \mapsto c(t)$ in S is denoted by D_c , $\frac{D}{dt}$ or simply $'$ if it is not necessary to specify the curve; if $V(t)$ is a vector field along c that extends to a vector field \widehat{V} on S , we have $V'(t) = \nabla_{\dot{c}(t)} \widehat{V}$.

Given a smooth map $(s, t) \mapsto \alpha(s, t) \in S$, we let $\frac{D}{\partial s}$ denote covariant differentiation along a curve of the form $s \mapsto \alpha(s, t)$ for a fixed t . Similarly $\frac{D}{\partial t}$ denotes covariant differentiation along a curve of the form $t \mapsto \alpha(s, t)$ for a fixed s . The symmetry of the Levi-Civita connection means that

$$\frac{D}{\partial s} \frac{\partial \alpha}{\partial t}(s, t) = \frac{D}{\partial t} \frac{\partial \alpha}{\partial s}(s, t)$$

for all s and t .

A geodesic segment $\gamma: I \rightarrow S$ is a curve satisfying $\gamma''(t) = D_\gamma \dot{\gamma}(t) = 0$, for all $t \in I$. Throughout this paper, all geodesics are assumed to be unit speed: $\|\dot{\gamma}\| \equiv 1$.

The Riemannian curvature tensor R is defined by

$$R(A, B)C = (\nabla_A \nabla_B - \nabla_B \nabla_A - \nabla_{[A, B]})C$$

and the Gaussian curvature $K: S \rightarrow \mathbb{R}$ is defined by

$$K(p) = \langle R(v, Jv)Jv, v \rangle,$$

where $v \in T_p^1 S$ is an arbitrary unit vector.

For $v \in TS$, we represent each element $\xi \in T_v TS$ in the standard way as a pair $\xi = (v_1, v_2)$ with $v_1 \in T_p S$ and $v_2 \in T_p S$, as follows. Each element $\xi \in T_v TS$ is tangent to a curve $V: (-1, 1) \rightarrow T^1 S$ with $V(0) = v$. Let $c = \pi \circ V: (-1, 1) \rightarrow S$ be the curve of basepoints of V in S , where $\pi: TS \rightarrow S$ is the standard projection. Then ξ is represented by the pair

$$(\dot{c}(0), D_c V(0)) \in T_p S \times T_p S.$$

Regarding TTS as a bundle over S in this way gives rise to a natural Riemannian metric on TS , called the *Sasaki metric*. In this metric, the inner product of two elements (v_1, w_1) and (v_2, w_2) of $T_v TS$ is defined:

$$\langle (v_1, w_1), (v_2, w_2) \rangle_{Sas} = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle.$$

This metric is induced by a symplectic form $d\omega$ on TTS ; for vectors (v_1, w_1) and (v_2, w_2) in $T_v TS$, we have:

$$d\omega((v_1, w_1), (v_2, w_2)) = \langle v_1, w_2 \rangle - \langle w_1, v_2 \rangle.$$

This symplectic form is the pull back of the canonical symplectic form on the cotangent bundle T^*S by the map from TS to T^*S induced by identifying a vector $v \in T_p S$ with the linear function $\langle v, \cdot \rangle$ on $T_p S$.

1.1. The geodesic flow and Jacobi fields. For $v \in TS$ let γ_v denote the unique geodesic γ_v satisfying $\dot{\gamma}_v(0) = v$. The geodesic flow $\varphi_t: TS \rightarrow TS$ is defined by

$$\varphi_t(v) = \dot{\gamma}_v(t),$$

wherever this is well-defined. The geodesic flow is always defined locally.

The *geodesic spray* is the vector field $\dot{\varphi}$ on TS (that is, a section of TTS) generating the geodesic flow. In the natural coordinates on TTS given by the connection, we have $\dot{\varphi}(v) = (v, 0)$, for each $v \in TS$. The spray is tangent to the level sets $\|\cdot\| = \text{const}$. Henceforth when we refer to the geodesic flow φ_t , we implicitly mean the restriction of this flow to the unit tangent bundle $T^1 S$.

Since the geodesic flow is Hamiltonian, it preserves a natural volume form on $T^1 S$ called the Liouville volume form. When the integral of this form is finite, it induces a unique probability measure on $T^1 S$ called the *Liouville measure* or *Liouville volume*.

Consider now a one-parameter family of geodesics, that is a map $\alpha: (-1, 1)^2 \rightarrow S$ with the property that $\alpha(s, \cdot)$ is a geodesic for each $s \in (-1, 1)$. Denote by $\mathcal{J}(t)$ the vector field

$$\mathcal{J}(t) = \frac{\partial \alpha}{\partial s}(0, t)$$

along the geodesic $\gamma(t) = \alpha(0, t)$. Then \mathcal{J} satisfies the *Jacobi equation*:

$$(1) \quad \mathcal{J}'' + R(\mathcal{J}, \dot{\gamma})\dot{\gamma} = 0,$$

in which $'$ denotes covariant differentiation along γ . Since this is a second order linear ODE, the pair of vectors $(\mathcal{J}(0), \mathcal{J}'(0)) \in T_{\gamma(0)} M \times T_{\gamma(0)} M$ uniquely determines the vectors $\mathcal{J}(t)$ and $\mathcal{J}'(t)$ along $\gamma(t)$. A vector field \mathcal{J} along a geodesic γ satisfying the Jacobi equation is called a *Jacobi field*.

The pair $(\mathcal{J}(t), \mathcal{J}'(t))$ corresponds in the manner described above to the tangent vector at $s = t$ to the curve $s \mapsto \frac{\partial \alpha}{\partial t}(s, t) = \varphi_t \circ V(s)$, which is $D\varphi_t(V'(0))$. Thus

Proposition 1.1. *The image of the tangent vector $(v_1, v_2) \in T_v TS$ under the derivative of the geodesic flow $D_v \varphi_t$ is the tangent vector $(\mathcal{J}(t), \mathcal{J}'(t)) \in T_{\varphi_t(v)} TS$, where \mathcal{J} is the unique Jacobi field along γ satisfying $\mathcal{J}(0) = v_1$ and $\mathcal{J}'(0) = v_2$.*

Computing the Wronskian of the Jacobi field $\dot{\gamma}$ and an arbitrary Jacobi field \mathcal{J} shows that $\langle \mathcal{J}', \dot{\gamma} \rangle$ is constant. It follows that if $\mathcal{J}'(t_0) \perp \dot{\gamma}(t_0)$ for some t_0 , then $\mathcal{J}'(t) \perp \dot{\gamma}(t)$ for all t . Similarly if $\mathcal{J}(t_0) \perp \dot{\gamma}(t_0)$ and $\mathcal{J}'(t_0) \perp \dot{\gamma}(t_0)$ for some t_0 , then $\mathcal{J}(t) \perp \dot{\gamma}(t)$ and $\mathcal{J}'(t) \perp \dot{\gamma}(t)$ for all t ; in this case we call \mathcal{J} a *perpendicular Jacobi field*. If α is a variation of geodesics giving rise to a perpendicular Jacobi field, then we call α a *perpendicular variation of geodesics*.

The space of all perpendicular Jacobi fields along a unit speed geodesic γ corresponds to the orthogonal complement $\dot{\varphi}^\perp(v)$ (in the Sasaki metric) to the geodesic spray $\dot{\varphi}(v)$ at the point $v = \dot{\gamma}(0) \in T^1 S$. To estimate the norm of the derivative $D\varphi_t$ on $TT^1 S$, it suffices to restrict attention to vectors in the invariant subspace $\dot{\varphi}^\perp$; that is, it suffices to estimate the growth of perpendicular Jacobi fields along unit speed geodesics.

Because S is a surface, the Jacobi equation (1) of a perpendicular Jacobi field along a unit speed geodesic segment can be expressed as a scalar ODE in one variable. Given such a geodesic $\gamma: I \rightarrow S$, any perpendicular Jacobi field \mathcal{J} along γ can be written in the form $(\mathcal{J}(t), \mathcal{J}'(t)) = (j(t)J\dot{\gamma}(t), j'(t)J\dot{\gamma}(t))$, where $j: I \rightarrow \mathbb{R}$ satisfies the *scalar Jacobi equation*:

$$(2) \quad j''(t) = -K(\gamma(t))j(t).$$

To analyze solutions to (2) it is often convenient to consider the functions $u(t) = j'(t)/j(t)$ and $\zeta(t) = j(t)/j'(t)$ which satisfy the Riccati equations $u'(t) = -K(\gamma(t)) - u^2(t)$ and $\zeta'(t) = 1 + K(\gamma(t))\zeta^2(t)$, respectively. In the next subsection, we describe some techniques for analyzing solutions to these types of equations.

1.2. Comparison lemmas for Ordinary Differential Equations. We will use a few basic comparison lemmas for solutions to ordinary differential equations. The first is standard and is presented without proof:

Lemma 1.2. [Basic comparison] *Let $F: \mathbb{R} \times [t_0, t_1] \rightarrow \mathbb{R}$ be C^1 , and let $\zeta: [t_0, t_1] \rightarrow \mathbb{R}$ be a solution to the ODE*

$$(3) \quad \zeta'(t) = F(\zeta(t), t).$$

Suppose that $\underline{u}, \bar{u}: [t_0, t_1] \rightarrow \mathbb{R}$ are C^1 functions satisfying $\underline{u}(t_0) \leq \zeta(t_0) \leq \bar{u}(t_0)$. Then the following hold:

- *If $F(\bar{u}(t), t) \leq \bar{u}'(t)$ for all $t \in [t_0, t_1]$, then $\zeta(t) \leq \bar{u}(t)$ for all $t \in [t_0, t_1]$.*
- *If $F(\bar{u}(t), t) < \bar{u}'(t)$ for all $t \in [t_0, t_1]$, then $\zeta(t) < \bar{u}(t)$ for all $t \in (t_0, t_1]$.*
- *If $F(\underline{u}(t), t) \geq \underline{u}'(t)$ for all $t \in [t_0, t_1]$, then $\zeta(t) \geq \underline{u}(t)$ for all $t \in [t_0, t_1]$.*
- *If $F(\underline{u}(t), t) > \underline{u}'(t)$ for all $t \in [t_0, t_1]$, then $\zeta(t) > \underline{u}(t)$ for all $t \in (t_0, t_1]$.*

We will have several occasions to deal with Riccati equations of the form $\zeta'(t) = 1 - k^2(t)\zeta^2(t)$ on an interval $[t_0, t_1]$ (where typically $-k^2(t) = K(\gamma(t))$ for some geodesic segment γ). Since the curvature of the surfaces we consider is not bounded away from $-\infty$, most of the ODEs we deal with will have unbounded coefficients. This necessitates reproving some standard results about solutions. A key basic result is the following.

Lemma 1.3. [Existence of Unstable Riccati Solutions] *Suppose $k: (t_0, t_1] \rightarrow \mathbb{R}_{>0}$ is a C^1 function satisfying $\lim_{t \rightarrow t_0} k(t) = \infty$. Then there exists a unique solution $\zeta(t)$ to the Riccati equation*

$$(4) \quad \zeta' = 1 - k^2 \zeta^2$$

for $t \in (t_0, t_1]$ satisfying $\zeta(t) > 0$ on $(t_0, t_1]$ and $\lim_{t \rightarrow t_0} \zeta(t) = 0$.

Moreover, if $\underline{k}: (t_0, t_1] \rightarrow \mathbb{R}_{>0}$ is any C^1 function satisfying $\underline{k}'(t) < 0$ and $\underline{k}(t) \leq k(t)$, for all $t \in (t_0, t_1]$, then $\zeta(t) \leq \underline{k}(t)^{-1}$, for all $t \in [t_0, t_1]$.

Proof. Let \underline{k} be a function satisfying the hypotheses of the lemma. Then there is a function $k_0: (t_0, t_1] \rightarrow \mathbb{R}_{>0}$ such that $k_0'(t) < 0$ and $\underline{k}(t) \leq k_0(t) \leq k(t)$ for all $t \in (t_0, t_1]$ and $k_0(t)^{-1} \rightarrow 0$ as $t \rightarrow t_0$. Observe that $(d/dt)(k_0^{-1}(t)) > 0 \geq 1 - k(t)^2 k_0(t)^{-2}$ for $t_0 < t \leq t_1$.

Now fix a decreasing sequence $t_n \rightarrow t_0$ in (t_0, t_1) . For each $n > 1$ let ζ_n be the solution to (4) on $[t_n, t_1]$ with $\zeta_n(t_n) = 0$. We can apply Lemma 1.2 to equation (4) on the interval $[t_n, t_1]$ with $\underline{u}(t) = 0$ and $\bar{u}(t) = k_0(t)^{-1}$. This gives us $0 < \zeta_n(t) \leq k_0(t)^{-1}$ for $t_n < t \leq t_1$. We can also apply Lemma 1.2 on this interval with $\zeta = \zeta_m$ for $m \geq n$ and $\underline{u} = \zeta_n$. This gives us $\zeta_m(t) \geq \zeta_n(t)$, for $t_n \leq t \leq t_1$.

The sequence of solutions ζ_n is thus increasing, positive and bounded above by k_0^{-1} . It follows that the function $\zeta := \lim_{n \rightarrow \infty} \zeta_n$ is a solution to (4), is positive on $(t_0, t_1]$, is bounded above by k_0^{-1} , and thus satisfies $\lim_{t \rightarrow t_0} \zeta(t) = 0$.

It remains to show that ζ is the only solution of (4) with the desired properties. Suppose η is another such solution. Since the graphs of two solutions of (4) cannot cross, we may assume that $\zeta(t) \geq \eta(t) \geq 0$ for $t_0 \leq t \leq t_1$. But then

$$(\zeta - \eta)'(t) = k(t)^2[(\eta(t)^2 - \zeta(t)^2)] \leq 0$$

for $t_0 < t \leq t_1$. Since $(\zeta - \eta)(t) \rightarrow 0$ as $t \rightarrow t_0$, this is possible only if $\zeta(t) = \eta(t)$ for $t_0 \leq t \leq t_1$. \diamond

We call the solution of the Riccati equation defined by the previous lemma the *unstable solution* on $(t_0, t_1]$.

Lemma 1.4. [Comparison of Unstable Riccati Solutions] *For $i = 1, 2$, let $k_i: (t_0, t_1] \rightarrow \mathbb{R}_{>0}$ be a C^1 function satisfying $\lim_{t \rightarrow t_0} k_i(t) = \infty$ and let $\zeta_i: (t_0, t_1] \rightarrow \mathbb{R}_{>0}$ be the unstable solution. Suppose $k_1(t) \geq k_2(t)$ for all $t \in (t_0, t_1]$. Then $\zeta_1(t) \leq \zeta_2(t)$ for all $t \in [t_0, t_1]$.*

Proof. Suppose $\zeta_2(t'_0) \geq \zeta_1(t'_0)$ for some $t'_0 \in (t_0, t_1]$. Then we can apply Lemma 1.2 to the equation $\zeta' = 1 - k_1^2 \zeta^2$ with $\zeta = \zeta_1$ and $\bar{u} = \zeta_2$ to obtain $\zeta_2(t) \geq \zeta_1(t)$ for all $t \in [t'_0, t_1]$. It now suffices to show that if there is $t'_1 \in (t_0, t_1]$ such that $\zeta_1(t) \geq \zeta_2(t)$ for all $t \in [t_0, t'_1]$, then we must have $\zeta_1(t) = \zeta_2(t)$ for all $t \in [t_0, t'_1]$. But if $\zeta_1 \geq \zeta_2 \geq 0$ on $(t_0, t'_1]$ we have

$$(\zeta_1 - \zeta_2)'(t) = k_2(t)^2 \zeta_2(t)^2 - k_1(t)^2 \zeta_1(t)^2 \leq 0$$

for $t_0 < t \leq t'_1$. Since $(\zeta_1 - \zeta_2)(t) \rightarrow 0$ as $t \rightarrow t_0$, this is possible only if $\zeta_1(t) = \zeta_2(t)$ for $t_0 \leq t \leq t'_1$. \diamond

Lemma 1.5. *Let $k: (0, t_1] \rightarrow \mathbb{R}_{>0}$ be a C^1 function satisfying $\lim_{t \rightarrow 0} k(t) = \infty$, and let $\zeta(t)$ be the unstable solution. Let $r > 1$.*

- (1) If $k(t)^2 \geq r(r-1)/t^2$ for all $t \in (0, t_1]$, then $\zeta(t) \leq t/r$ for all $t \in (0, t_1]$.
- (2) If $k(t)^2 \leq r(r-1)/t^2$ for all $t \in (0, t_1]$, then $\zeta(t) \geq t/r$ for all $t \in (0, t_1]$.
- (3) Suppose $N > 0$, $0 < t_2 < \min\{t_1, (r-1)/N\}$ and

$$k(t)^2 \in \left[\frac{r(r-1)}{t^2} - \frac{N}{t}, \frac{r(r-1)}{t^2} + \frac{N}{t} \right] \quad \text{for all } t \in (0, t_2].$$

Then there exists $M > 0$ such that

$$\zeta(t) \in \left[\frac{t}{r} - Mt^2, \frac{t}{r} + Mt^2 \right] \quad \text{for all } t \in (0, t_2].$$

Proof. 1 and 2. These follow from Lemma 1.4 because $\zeta(t) = t/r$ is the unstable solution for

$$\zeta' = 1 - \frac{r(r-1)}{t^2} \zeta^2.$$

3. Choose $\delta > 0$ such that $0 < t_2 < 1/2\delta < (r-1)/N$. Then,

$$\frac{(r-\delta t)(r-\delta t-1)}{t^2} \leq \frac{r(r-1)}{t^2} - \frac{N}{t} \quad \text{and} \quad \frac{(r+\delta t)(r+\delta t-1)}{t^2} \geq \frac{r(r+1)}{t^2} + \frac{N}{t}$$

for $0 < t \leq t_2$. It follows from parts 1 and 2 of this lemma that for each $\tau \in (0, t_2]$ we have $\zeta(t) \in [t/(r+\delta\tau), t/(r-\delta\tau)]$, for all $t \in (0, \tau]$. Consequently, $\zeta(t) \in [t/(r+\delta t), t/(r-\delta t)]$, for all $t \in (0, t_2]$. Now choose $M > 2\delta/r$. We then have

$$\frac{t}{r} - Mt^2 \leq \frac{t}{(r+\delta t)} \quad \text{and} \quad \frac{t}{(r-\delta t)} \leq \frac{t}{r} + Mt^2,$$

for $0 < t \leq t_2$. We conclude that $\zeta(t) \in [t/r - Mt^2, t/r + Mt^2]$, for $0 \leq t \leq t_2$. \diamond

2. REGULARITY OF THE DISTANCE δ TO THE CUSP

Suppose S satisfies the hypotheses of Theorem 1, with $k = 1$. Before considering the global properties of the metric on S , we introduce local coordinates about the puncture p_1 and study the behavior of geodesics that remain in this cuspidal region during some time interval.

In this section and the next, we thus assume that the punctured disk \mathbb{D}^* has been endowed with an incomplete Riemannian metric, whose completion is the closed disk $\overline{\mathbb{D}}$. Assume that the lift of this metric to $\widetilde{\mathbb{D}^*}$ is *geodesically convex*: that is, any two points in $\widetilde{\mathbb{D}^*}$ can be connected by a unique geodesic in $\widetilde{\mathbb{D}^*}$.

Let ρ be the Riemannian distance metric on $\overline{\mathbb{D}}$ and for $z \in \mathbb{D}$, let $\delta(z) = \rho(z, 0)$. For $\delta_0 \in (0, 1)$, we denote by $\mathbb{D}^*(\delta_0)$ the set of $z \in \mathbb{D}^*$ with $\delta(z) < \delta_0$.

Assume that there exists $r > 2$ such that for all $z \neq 0$ the curvature of the Riemannian metric satisfies:

$$(5) \quad 0 > K(z) = -\frac{r(r-1)}{\delta(z)^2} + O(\delta(z)^{-1}),$$

and

$$(6) \quad \|\nabla^j K(z)\| = O(\delta(z)^{-2-j}),$$

for $j = 1, 2, 3$.

The main result of this section establishes regularity of the function δ and estimates on the size of its derivatives. We also introduce a function c that measures the geodesic

curvature of the level sets of δ and establish some of its properties. The results in this section establish in this incomplete, singular setting the standard regularity properties of Busemann functions for compact, negatively curved manifolds (see, e.g. [12]) – in particular, Busemann functions for a C^k metric are C^{k-1} . The main techniques are thus fairly standard but require some care in the use of comparison lemmas for ODEs. To avoid tedium, we have described many calculations in detail but have left others to the reader.

Proposition 2.1. *The cusp distance function δ is C^4 . Let $V = \nabla\delta$, and let $c: \mathbb{D}^* \rightarrow \mathbb{R}_{>0}$ be the geodesic curvature function defined by*

$$(7) \quad c = \langle \nabla_{JV} V, JV \rangle.$$

Then:

- (1) $\nabla_{JV} V = [JV, V] = cJV$.
- (2) *for any vector field U :* $\nabla_U V = c\langle U, JV \rangle JV$, and $\nabla_U JV = -c\langle U, JV \rangle V$.
- (3) $c = r/\delta + O(1)$.
- (4) $\|\nabla c\| = O(\delta^{-2})$.
- (5) $\|\nabla^2 c\| = O(\delta^{-3})$.

Corollary 2.2. *The function δ satisfies: $\|\nabla\delta\| = 1$ and $\|\nabla^i\delta\| = O(\delta^{1-i})$, for $i = 2, 3, 4$.*

Proof. This follows from the facts: $\nabla_U \delta = \langle U, V \rangle$, $\nabla_U V = c\langle U, JV \rangle JV = O(\delta^{-1})\|U\|$, and $\|\nabla^j c\| = O(\delta^{-1-j})$, $j = 1, 2$, proved in Proposition 2.1. \diamond

Proof of Proposition 2.1. We prove first that δ is C^4 , in several steps.

Step 0: δ is continuous. We realize the universal cover of the punctured disk \mathbb{D}^* as the strip $\mathbb{R} \times (0, 1)$ with the deck transformations $(x, t) \mapsto (x + n, t)$, $n \in \mathbb{Z}$. Endow $\mathbb{R} \times (0, 1)$ with the lifted metric, which is geodesically convex by assumption, and lift δ to a function $\tilde{\delta}$. By assumption, the completion of \mathbb{D}^* is $\overline{\mathbb{D}}$, and so the completion $\overline{\mathbb{R} \times (0, 1)}$ in this metric is the union of $\mathbb{R} \times (0, 1]$ with a single point p^* .

Since $\mathbb{R} \times (0, 1)$ is negatively curved and geodesically convex, it is in particular $CAT(0)$. The $CAT(0)$ property is preserved under completion, and so $\overline{\mathbb{R} \times (0, 1]}$ is also $CAT(0)$. Thus for every $\tilde{z}_0 \in \mathbb{R} \times (0, 1]$, there is a unique unit-speed geodesic from \tilde{z}_0 to p^* . This projects to a (unique) geodesic in \mathbb{D} from z_0 to 0.

Fix $z_0 \in \mathbb{D}^*$ with lift $\tilde{z}_0 \in \mathbb{R} \times (0, 1]$, and let $\gamma_{0, z_0}: [0, \delta(z_0)] \rightarrow \overline{\mathbb{D}}$ be the unit-speed geodesic from 0 to z_0 found by the previous argument. It has the property that $\delta(\gamma_{0, z_0}(t)) = \rho(0, \gamma_{0, z_0}(t)) = t$ for every $t \in [0, \delta(z_0)]$. Let $t_n \rightarrow 0$ be a sequence of times in $(0, \delta(z_0))$, and define a sequence of functions $\tilde{\delta}_n: \mathbb{R} \times (0, 1] \rightarrow \mathbb{R}_{>0}$ by $\delta_n(\tilde{z}) = \tilde{\rho}(\tilde{z}, \tilde{\gamma}_{0, \tilde{z}_0}(t_n))$. The $\tilde{\delta}_n$ are convex, C^3 away from $\tilde{\gamma}_{0, \tilde{z}_0}(t_n)$, and $\|\nabla\delta_n\| = 1$ for all n .

Lemma 2.3. *For every $\tilde{z} \in \mathbb{R} \times (0, 1]$ and all $m \geq n$, we have*

$$(8) \quad |\tilde{\delta}_n(\tilde{z}) - \tilde{\delta}_m(\tilde{z})| \leq t_n - t_m \leq t_n,$$

Proof. This follows from the triangle inequality. \diamond

Since $\tilde{\delta}_n$ is Cauchy, it converges (locally uniformly in $\mathbb{R} \times (0, 1]$) to a continuous, convex function $\hat{\delta}$. Moreover $\hat{\delta}(\tilde{z})$ is the distance $\tilde{\rho}(\tilde{z}, p^*)$. It follows that $\tilde{\delta}$ is continuous (and convex), and so δ is continuous.

Step 1: δ is C^1 . Let $\tilde{V}_n = \nabla \tilde{\delta}_n$ be the corresponding sequence of radial vector fields on $\mathbb{R} \times (0, 1]$.

Lemma 2.4. Fix $\tilde{z} \in \mathbb{R} \times (0, 1]$. For all $m \geq n$ sufficiently large, we have:

$$\|\tilde{V}_n(\tilde{z}) - \tilde{V}_m(\tilde{z})\| \leq \frac{t_n}{\tilde{\delta}(\tilde{z}) - t_n}.$$

Thus \tilde{V}_n is a Cauchy sequence in the local uniform topology.

Proof. This is a standard argument in negative curvature (in fact nonpositive curvature suffices). This uses that $\|\nabla \delta_n\| = 1$ for all n . \diamond

This lemma implies that δ is C^1 . Let \tilde{V} be the local uniform limit of the \tilde{V}_n : by definition, $V = \nabla \tilde{\delta}$. Let $V = \nabla \delta$ be the projection of \tilde{V} to \mathbb{D}^* . It remains to show that V is C^3 , which implies that δ is C^4 .

Step 2: V is C^1 .

Lemma 2.5. There exists $\delta_0 > 0$, such that for every $z_0 \in \mathbb{D}^*$ with $\delta(z_0) < \delta_0$, the following holds. For every vector field U , $\nabla_U V$ exists and is continuous in a neighborhood of $\gamma_{0,z_0}((0, \delta(z_0)))$. Moreover:

$$(9) \quad \nabla_{JV} V(\gamma_{0,z_0}(t)) = \zeta(t)^{-1} JV(\gamma_{0,z_0}(t)),$$

for all $t \in (0, \delta(z_0)]$, where ζ is the positive solution to the Riccati equation

$$(10) \quad \zeta'(t) = 1 + K(\gamma_{0,z_0}(t))\zeta(t)^2,$$

given by Lemma 1.3, satisfying $\zeta(0) = 0$.

Proof. Fix $\delta_0 > 0$ (we will specify how small it must be later). Fix z_0 with $\delta(z_0) \leq \delta_0$, and denote by γ the geodesic γ_{0,z_0} .

For each n , define a perpendicular, radial variation of geodesics $\gamma_n(s, t)$ by the properties: $\gamma_n(0, t) = \gamma(t)$, $\gamma_n(s, t_n) = \gamma(t_n)$ and

$$\frac{D^2 \gamma_n}{\partial s \partial t}(s, \delta_0) = J \dot{\gamma}_n(s, \delta_0),$$

for all s, t with $t \geq t_n$ (and s belonging to a small, fixed neighborhood of 0). Let $\delta_n: \mathbb{D}^* \rightarrow \mathbb{R}_{>0}$ be defined by $\delta_n(z) = \rho(z, \gamma(t_n))$; then for $t \geq t_n$, and s sufficiently small, we have

$$\delta_n(\gamma_n(s, t)) = t - t_n.$$

It follows that for any $\epsilon > 0$, if n sufficiently large and $t \geq (1 + 2/\epsilon)t_n$, we have

$$(11) \quad \delta(\gamma_n(s, t)) \in [(1 - \epsilon)t + \epsilon t_n, t].$$

We have already shown (working on the universal cover) that in a neighborhood of γ , we have $\delta_n \rightarrow \delta$ and $V_n = \nabla \delta_n \rightarrow V$ uniformly on compact sets. Let $\gamma(s, t)$ be the limiting variation of geodesics, which satisfies $\gamma(0, t) = \gamma(t)$, and $\delta(\gamma(s, t)) = t$. At this point we have shown that γ is C^1 , with $\partial \gamma / \partial s(s, \delta_0) = JV(\gamma(s, \delta_0))$. Note that $V_n(\gamma_n(0, t)) = V(\gamma_n(0, t)) = V(\gamma(t))$, for all $n, t \geq t_n$.

Since $V_n \rightarrow V$, it suffices to show that

$$\nabla_{V_n} V_n(\gamma_n(s, t)) \rightarrow \nabla_V V(\gamma(s, t)) \text{ and } \nabla_{JV_n} V_n(\gamma_n(s, t)) \rightarrow \nabla_{JV} V(\gamma(s, t)).$$

The proof that $\nabla_{V_n} V_n(\gamma_n(s, t)) \rightarrow \nabla_V V(\gamma(s, t))$ is immediate: since γ_n is a variation of geodesics, we have $\gamma_n'' = \nabla_{V_n} V_n \equiv 0 \equiv \nabla_V V = \gamma''$.

We now show that $\nabla_{JV_n} V_n(\gamma_n(s, t)) \rightarrow \nabla_{JV} V(\gamma(s, t))$. Let $j_n(s, t)$ be the scalar Jacobi field associated with the perpendicular variation γ_n :

$$\frac{\partial \gamma_n}{\partial s} = j_n(s, t) J \dot{\gamma}_n(s, t) = j_n(s, t) J V_n(s, t).$$

On the one hand,

$$\frac{D}{\partial s} V_n(\gamma_n(s, t)) = \frac{D^2}{\partial s \partial t} \gamma_n(s, t) = j_n'(s, t) J V_n(s, t),$$

while on the other hand,

$$\frac{D}{\partial s} V_n(\gamma_n(s, t)) = \nabla_{j_n(s, t) J V_n(s, t)} V_n = j_n(s, t) \nabla_{J V_n} V_n(\gamma_n(s, t)).$$

Writing $\zeta_n(s, t) = j_n(s, t)/j_n'(s, t)$, we thus have $\zeta_n(s, t) \nabla_{J V_n} V_n(\gamma_n(s, t)) = J V_n(\gamma_n(s, t))$. We prove that $\zeta_n(s, t)$ converges to $\zeta(s, t)$, the positive solution to (10).

To see this, we first establish uniform upper and lower bounds for $\zeta_n(s, t)$, for $t \geq (1 + 2/\epsilon)t_n$. The ζ_n satisfy the Riccati equation:

$$(12) \quad \zeta_n'(s, t) = 1 + K(\gamma_n(s, t)) \zeta_n(s, t)^2,$$

with $\zeta_n(s, t_n) = 0$, for all s . Now

$$K(\gamma_n(s, t)) = -\frac{r(r-1)}{\delta(\gamma_n(s, t))^2} + O(\delta(\gamma_n(s, t))^{-1}).$$

By (11) we thus have:

$$(13) \quad K(\gamma_n(s, t)) \in \left[-\frac{r^2}{t^2}, -\frac{(r-1)^2}{t^2} \right],$$

if $\delta(z_0) \leq \delta_0$ is sufficiently small, $t \geq (1 + 2/\epsilon)t_n$, and n is sufficiently large.

We show that there exists $\mu \in (0, 1)$ such that, for n sufficiently large, we have

$$(14) \quad \zeta_n(t) \in [\mu(t - t_n), \mu^{-1}(t - t_n)].$$

To see the lower bound, let $u = \mu(t - t_n)$. Then $u' = \mu$. On the other hand, when $\zeta_n(t) = u$, we have $\zeta_n' = 1 + K\mu^2(t - t_n)^2$. This is larger than u' provided that $1 + K\mu^2(t - t_n)^2 > \mu$. But this will hold if $1 - r^2\mu^2 > \mu$. The upper bound is similarly obtained. By Lemma 1.2, $\zeta_n(t) \geq \mu(t - t_n)$ for all $t \geq t_n$, which proves (14).

We now use Lemma 1.2 to prove that for some large but fixed C :

$$(15) \quad |(\zeta_m - \zeta_n)(s, t)| \leq C t_n,$$

for all s and $t \geq (1 + 2/\epsilon)t_n$. For $m \geq n$, we have, since $\zeta_n(s, t_n) = 0$:

$$|(\zeta_m - \zeta_n)(s, t_n)| \leq |\zeta_m(s, t_n)| + |\zeta_n(s, t_n)| = |\zeta_m(s, t_n)| \leq t_n.$$

Subtracting the ODEs for ζ_m and ζ_n , we have for $t > t_n$:

$$\begin{aligned} (\zeta_n - \zeta_m)'(s, t) &= K(\gamma_n(s, t)) \zeta_n(s, t)^2 - K(\gamma_m(s, t)) \zeta_m(s, t)^2 = \\ &= (K(\gamma_n(s, t)) - K(\gamma_m(s, t))) \zeta_m(s, t)^2 + K(\gamma_n(s, t)) (\zeta_n(s, t)^2 - \zeta_m(s, t)^2) \\ &= O\left(\frac{t_n(t - t_n)^2}{t^3}\right) + K(\gamma_n(s, t))(\zeta_n(s, t) + \zeta_m(s, t))(\zeta_n(s, t) - \zeta_m(s, t)), \end{aligned}$$

since $\|\nabla K(\gamma_n(s, t))\| = O(\delta(\gamma_n(s, t))^{-3}) = O(t^{-3})$, $\rho(\gamma_n(s, t), \gamma_m(s, t)) = O(t_n)$, and $\zeta_m(s, t) = O(t - t_m)$. Writing $y = \zeta_n - \zeta_m$, we have that y satisfies the ODE

$$(16) \quad y' = O\left(\frac{t_n(t - t_n)^2}{t^3}\right) + K(\gamma_n(s, t))(\zeta_n(s, t) + \zeta_m(s, t))y.$$

Fix n and let $u(t) = Ct_n$. Then $u'(t) = 0$, and y' evaluated at $y = u$ is

$$y' = O\left(\frac{t_n(t - t_n)^2}{t^3}\right) + K(\gamma_n(s, t))(\zeta_n(s, t) + \zeta_m(s, t))Ct_n$$

We claim that if C is sufficiently large, then $y' \leq 0 = u'$. To see this fix $N > 0$ such that

$$y' \leq \frac{Nt_n(t - t_n)^2}{t^3} + K(\gamma_n(s, t))(\zeta_n(s, t) + \zeta_m(s, t))y.$$

Then (13) and (14) imply that

$$y' \leq \frac{Nt_n(t - t_n)^2}{t^3} - 2\frac{(r-1)^2}{t^2}\mu(t - t_n)Ct_n \leq \frac{t_n(t - t_n)}{t^2}(N - 2(r-1)^2\mu C).$$

Clearly this is ≤ 0 for $t \geq t_n$ if C is sufficiently large. Since $y(t_n) \leq t_n \leq u$, for all $C \geq 1$, this implies by Lemma 1.2 that $y \leq u$ for all $t \geq (1 + 2/\epsilon)t_n$; a similar argument shows that $y \geq -u$, and hence if C is sufficiently large and $m \geq n$, then for all $t \geq (1 + 2/\epsilon)t_n$ (15) holds.

Thus $|(\zeta_m - \zeta_n)(s, t)|$ tends to 0 as $t_n \rightarrow 0$, with s, t fixed. Thus the $\zeta_n(s, t)$ converge, and since they satisfy (12), their limit $\zeta(s, t)$ satisfies (10). We obtain that the functions $\zeta_n(s, t)\nabla_{JV_n}V_n(\gamma_n(s, t)) = JV_n(\gamma_n(s, t))$ converge locally uniformly, and hence $\nabla_{JV}V$ exists and is continuous.

Since $\nabla_{JV_n}V_n(\gamma_n(s, t)) = \zeta_n(s, t)^{-1}JV_n(\gamma_n(s, t))$, $\zeta_n \rightarrow \zeta$, and $JV_n \rightarrow JV$, we obtain (9) by taking a limit and setting $s = 0$. \diamond

In light of Lemma 2.5, we define a function $\nu: \mathbb{D}^* \rightarrow \mathbb{R}_{>0}$ as follows. For each $z \in \mathbb{D}^*$, let $\zeta: [0, \delta(z)] \rightarrow \mathbb{R}_{>0}$ be the positive solution to (10) given by Lemma 1.3. We then set $\nu(z) := \zeta(\delta(z))$. It follows immediately from Lemma 2.5 that for every $z \in \mathbb{D}^*$, we have

$$(17) \quad \nabla_{JV}V(z) = \nu(z)^{-1}JV(z).$$

Step 3: ν is C^1 .

To prove that δ is C^3 , it thus suffices to show that ν is C^1 . Equation (10) implies that $\nabla_V\nu$ exists and is continuous. It remains to show that $\nabla_{JV}\nu$ exists and is continuous.

We fix $\delta_0 > 0$ as above and let $z_0 \in \mathbb{D}^*(\delta_0)$, and we reintroduce the variations of geodesics $\gamma_n(s, t)$ defined by the properties: $\gamma_n(0, t) = \gamma_{0, z_0}(t)$, $\gamma_n(s, t_n) = \gamma_{0, z_0}(t_n)$ and

$$\frac{D^2\gamma_n}{\partial s \partial t}(s, \delta_0) = j'_n(s, \delta_0)J\dot{\gamma}_n(s, \delta_0),$$

for all s . As above, write $\gamma = \gamma_{0, z_0}$, and let $\gamma(s, t)$ be the limiting variation of geodesics. We observe that Lemma 2.5 also implies that $j_n \rightarrow j$ and $j'_n \rightarrow j'$, locally uniformly, where $\partial\gamma/\partial s = jJV$ and $D^2\gamma/\partial s \partial t = j'JV$. The convergence follows from the formulae:

$$j'_n(s, t) = \exp\left(\int_t^{\delta_0} K(\gamma_n(s, x))\zeta_n(x) dx\right), \quad j'(s, t) = \exp\left(\int_t^{\delta_0} K(\gamma(s, x))\zeta(x) dx\right),$$

$$j_n(s, t) = \zeta_n(s, t)j'_n(s, t), \quad \text{and} \quad j(s, t) = \zeta(s, t)j'(s, t).$$

Thus the variation γ is C^2 on \mathbb{D}^* , and satisfies $\gamma(s, 0) = 0$. We record here a lemma, which follows easily from these formulae, combined with (8), (13) and (15).

Lemma 2.6. *For all $t \geq t_n$, we have $j'_n(t)/j'_m(t) = 1 + O(t_n/t)$, and for all $t \geq (1 + 2/\epsilon)t_n$, we have $j_n(t)/j_m(t) = 1 + O(t_n/t)$.*

As above, let $\zeta_n(s, t)$ be the solution to (12), and let $\zeta(s, t)$ be the solution to (10). Note that $\nu(\gamma(s, t)) = \zeta(s, t)$.

To prove that $\nabla_{JV}\nu$ exists and is continuous, we show that $\partial_s \zeta_n(s, t)$ converges uniformly to $\partial_s \zeta(s, t)$, the unique bounded solution to

$$(18) \quad \partial_s \zeta'(s, t) = \partial_s K(\gamma(s, t))\zeta(s, t)^2 + 2\zeta(s, t)K(\gamma(s, t))\partial_s \zeta(s, t),$$

which satisfies $\partial_s \zeta(s, 0) = 0$, for all s . Since $\partial_s \zeta = j(s, t)\nabla_{JV}\zeta$ and $j_n \rightarrow j > 0$ locally uniformly, this will imply that $\nabla_{JV}\zeta$ exists and is continuous.

Lemma 2.7. *There exists $M > 0$ such for all $m \geq n$ and all $t \geq (1 + 2/\epsilon)t_n$, we have*

$$(19) \quad |\partial_s \zeta_n(s, t) - \partial_s \zeta_m(s, t)| \leq Mt_n j'_n(s, t).$$

Proof. Differentiating equation (12) with respect to s , we obtain:

$$(20) \quad \partial_s \zeta'_n(s, t) = \partial_s K(\gamma_n(s, t))\zeta_n(s, t)^2 + 2\zeta_n(s, t)K(\gamma_n(s, t))\partial_s \zeta_n(s, t).$$

Note that since $\zeta_n(s, t_n) = 0$, for all n , we have that $\partial_s \zeta_n(s, t_n) = 0$, for all n .

To simplify notation, fix s , and write $w_n(t) = \partial_s \zeta_n(s, t)$, $u_n(t) = \zeta_n(s, t)$, and $u = \zeta(s, t)$. Then $u_n(t_n) = 0$, for all n . From equations (14) and (15), we have $|u_n| \leq Ct$ and $|u_n - u_m| \leq Ct_n$. Then equation (20) gives

$$(21) \quad w'_n = j_n(\nabla_{JV_n} K)u_n^2 + 2u_n(K \circ \gamma_n)w_n$$

We first claim there exists $C > 0$ such that $|w_n(t)| \leq Cj_n(t)$, for all $t \geq t_n$. Let $y = Cj_n(t)$. Then $y' = Cj'_n(t)$, whereas w'_n evaluated at $w_n = y$ gives $w'_n = j_n(\nabla_{JV_n} K)u_n^2 + 2Cu_n(K \circ \gamma_n)j_n$. Then $w'_n \leq y'$ if and only if $j_n(\nabla_{JV_n} K)u_n^2 + 2Cu_n(K \circ \gamma_n)j_n \leq Cj'_n$; dividing through by j'_n , and recalling that $u_n = j_n/j'_n$, we are reduced to showing:

$$\nabla_{JV_n} K u_n^3 + 2Cu_n^2(K \circ \gamma_n) \leq C,$$

which holds if and only if $C(1 - 2u_n^2(K \circ \gamma_n)) \geq (\nabla_{JV_n} K)u_n^3$. Since $u_n = O(t)$, $K < 0$, and $\|\nabla_{JV_n} K\| = O(\delta^{-3}) = O(t^{-3})$, for n sufficiently large, this will hold provided that C and n are sufficiently large. We conclude that

$$(22) \quad |w_n(t)| \leq Cj_n(t),$$

for all $t \geq t_n$.

We next claim that there exists $M > 0$ such that for $m \geq n$ sufficiently large and $t \geq (1 + 2/\epsilon)t_n$, we have

$$(23) \quad |w_n(t) - w_m(t)| \leq Mt_n j'_n(t)$$

This will give the conclusion (19) of Lemma 2.7.

For $m \geq n$, subtracting the corresponding equations in (21), we obtain:

$$(24) \quad (w_n - w_m)' = j_n(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_m} K)u_m^2 + 2u_n(K \circ \gamma_n)w_n - 2u_m(K \circ \gamma_m)w_m.$$

We claim that there exists $N > 0$ such that for $m \geq n$, and $t \geq (1 + 2/\epsilon)t_n$, we have

$$(25) \quad |(w_n - w_m)' - 2u_n(K \circ \gamma_n)(w_n - w_m)| \leq \frac{Nt_n j_n}{t^2}.$$

Assuming this claim, let us complete the proof of (23). Let N be given so that (25) holds for $m \geq n$, and $t \geq (1+2/\epsilon)t_n$. Let $y(t) = w_n(t) - w_m(t)$. Then $|y' - 2u_n(K \circ \gamma_n)y| \leq \epsilon j_n/t^2$, and (22) implies that

$$\begin{aligned} y((1+2/\epsilon)t_n) &= |w_n((1+2/\epsilon)t_n) - w_m((1+2/\epsilon)t_n)| \leq |w_m((1+2/\epsilon)t_n)| + |w_n((1+2/\epsilon)t_n)| \\ &\leq C(j_m((1+2/\epsilon)t_n) + j_n((1+2/\epsilon)t_n)) \\ &\leq C(j'_m((1+2/\epsilon)t_n)u_m((1+2/\epsilon)t_n) + j'_n((1+2/\epsilon)t_n)u_n((1+2/\epsilon)t_n)) \\ &< Nt_n j'_n((1+2/\epsilon)t_n), \end{aligned}$$

for some $N > 0$, since $u_m((1+2/\epsilon)t_n), u_n((1+2/\epsilon)t_n) \leq 2\epsilon^{-1}\mu^{-1}t_n$, by (14), and $j'_n(t)/j'_m(t) = 1 + O(t_n/t)$, by Lemma 2.6. This shows that (23) holds at $t = (1+2/\epsilon)t_n$, provided n is sufficiently large.

We claim that there exists $M > 0$ such that for all such m, n , we have $|y(t)| \leq Mt_n j'_n(t)$, for $t \geq (1+2/\epsilon)t_n$. We prove the upper bound; the lower bound is similar. We will employ Lemma 1.2.

To this end, let $z(t) = Mt_n j'_n(t)$. Note that $z' = Mt_n j''_n = -Mt_n(K \circ \gamma_n)j_n$, whereas evaluating y' at $y = z$, we get

$$y'(t) \leq 2Mt_n u_n(K \circ \gamma_n)j'_n + \frac{Nt_n j_m}{t^2} = 2Mt_n(K \circ \gamma_n)j_n + \frac{Nt_n j_m}{t^2}.$$

To satisfy the hypotheses of Lemma 1.2, we require that $y'(t) \leq z'(t)$ whenever $y = z$, which is implied by:

$$2Mt_n(K \circ \gamma_n)j_n + \frac{Nt_n j_n}{t^2} \leq -Mt_n(K \circ \gamma_n)j_n,$$

or:

$$\frac{Nt_n j_n}{t^2} \leq -3Mt_n(K \circ \gamma_n)j_n.$$

Since $-K \circ \gamma_n(t) \geq (r-1)^2/t^2$ (by 13), we see that this will hold (for all n sufficiently large) if $M > N/3(r-1)^2$. This establishes (23).

We return to the proof of the claim that there exists an $N > 0$ such that for $m \geq n$, and $t \geq (1+2/\epsilon)t_n$ the inequality (25) holds. The proof amounts to adding and subtracting terms within the left hand side of (25), varying one at a time the multiplied quantity in each term. The added and subtracted terms are grouped in twos and the absolute value of the difference in each pair is bounded above. To illustrate, consider the difference appearing on the left hand side of (25). The first two terms appearing in that difference, coming from (24), are:

$$\begin{aligned} j_n(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_m} K)u_m^2 \\ = (j_n(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_n} K)u_n^2) + (j_m(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_m} K)u_m^2). \end{aligned}$$

The quantity $j_n(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_n} K)u_n^2$ can be bounded, and the remaining term $j_m(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_m} K)u_m^2$ can be further decomposed, as follows. First, using (14) to bound $|u_n|$, the assumption that $\|\nabla K\| = O(\delta^{-3})$ together with (11) to bound $\|\nabla_{JV_n} K\|$, and the fact from Lemma 2.6 that $(j_m - j_n) = j_n O(t_n/t)$, we have that

$$|j_n(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_n} K)u_n^2| = j_n u_n^2 |\nabla_{JV_n} K| \left| 1 - \frac{j_m}{j_n} \right| \leq j_n O\left(\frac{t_n}{t^2}\right).$$

Second, to deal with the remaining term $j_m(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_m} K)u_m^2$, we write:

$$j_m(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_m} K)u_m^2$$

$$= (j_m(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_n} K)u_m^2) + (j_m(\nabla_{JV_n} K)u_m^2 - j_m(\nabla_{JV_m} K)u_m^2),$$

and bound each term separately in a similar way to give a bound on the initial quantity $|j_n(\nabla_{JV_n} K)u_n^2 - j_m(\nabla_{JV_m} K)u_m^2|$ of order $j_n t_n / t^2$. The same procedure is used to bound the remaining part of the difference appearing in (25), which is:

$$\begin{aligned} & |(2u_n(K \circ \gamma_n)w_n - 2u_m(K \circ \gamma_m)w_m) - 2u_n(K \circ \gamma_n)(w_n - w_m)| \\ &= |-2u_m(K \circ \gamma_m)w_m + 2u_n(K \circ \gamma_n)w_m|. \end{aligned}$$

In all, we use the following bounds:

- δ is comparable to t , by (11);
- $u_n = O(t)$, by (14);
- $|u_n - u_m| = O(t_n)$, by (15);
- $|j_n - j_m| = O(j_n t_n / t)$, for $t \geq (1 + 2/\epsilon)t_n$, by Lemma 2.6;
- $|K \circ \gamma_n - K \circ \gamma_m| = O(t_n t^{-2})$, since $|K| = O(t^{-2})$ and $d(\gamma_n, \gamma_m) = O(t_n)$;
- $|\nabla_{JV_n} K - \nabla_{JV_m} K| = O(t_n t^{-3})$, which uses the bounds on $\|\nabla K\|$ and $\|\nabla^2 K\|$, as well as (11) and Lemma 2.4.

The details are left to the patient reader.

This finishes the verification of the claim in (25), and thus the proof of Lemma 2.7. \diamond

To finish the proof that ν is C^1 , note that equation (19) can then be re-expressed using (14):

$$(26) \quad |\partial_s \zeta_n(s, t) - \partial_s \zeta_m(s, t)| \leq M t_n \zeta_n^{-1}(s, t) j_n(s, t) \leq \frac{M t_n j_n(s, t)}{\mu(t - t_n)} \leq \frac{2 M t_n j_n(s, t)}{\mu t},$$

for $t \geq (1 + 2/\epsilon)t_n$. Recalling that $j_n \rightarrow j$, we conclude that $\partial_s \zeta_n(s, t) \rightarrow \partial_s \zeta(s, t)$ locally uniformly in s, t . The bounds $|\partial_s \zeta_n(s, t)| \leq C j_n(s, t)$ from (22) become in the limit $|\partial_s \zeta(s, t)| \leq C j(s, t)$. But $\partial_s \zeta(s, t) = j(s, t) \nabla_{JV} \zeta$, and so we conclude that

$$(27) \quad \|\nabla_{JV} \nu\| \leq C.$$

Step 4: ν is C^2 . A very similar proof to the one in Step 3 (with more terms to estimate, but using, in addition to the previously obtained bounds, the bound $\|\nabla^3 K\| = O(\delta^{-5})$) gives that ν is C^2 with $\|\nabla^2 \nu\| = O(\delta^{-1})$. One obtains this estimate as in the previous step by bounding $\|\nabla_V^2 \nu\|$, $\|\nabla_V \nabla_{JV} \nu\|$, $\|\nabla_{JV} \nabla_V \nu\|$, and $\|\nabla_{JV}^2 \nu\|$. Each of these is controlled by a differential equation, whose solutions can be estimated using a double variation of geodesics $\gamma(s_1, s_2, t)$. The key point, illustrated by the previous computations, is that because K has the “expected” order of derivatives with respect to δ , any quantity obtained by solving a first-order linear differential equation derived from the Riccati equation with coefficients expressed in terms of these derivatives will have the “expected” order in δ as well. Thus, $\|\nabla^i K\| = O(\delta^{-2-i})$ for $i = 1, 2, 3$ implies that $\|\nabla^i \nu\| = O(\delta^{1-i})$, for $i = 1, 2$.

This completes the proof that δ is C^4 . We now turn to items 1-5.

1. $\nabla_{JV} V - \nabla_V JV = [JV, V]$ by the symmetry of the Levi-Civita connection. But $\nabla_V JV = 0$.

2. For arbitrary U , we have $\nabla_U V = \langle U, V \rangle \nabla_V V + \langle U, JV \rangle \nabla_{JV} V = c \langle U, JV \rangle JV$, giving the first conclusion: $\nabla_U V = c \langle U, JV \rangle JV$. The second conclusion follows from the first and the fact that J is parallel.

3. Note that $\nu = c^{-1}$. It then is equivalent to prove that $\nu = \delta/r + O(\delta^2)$. Fix z and let $\gamma = \gamma_{0,z}$. Along this geodesic, $\zeta(t) = \nu(\gamma(t))$ satisfies the equation (10). On the other hand $-K(\gamma(t)) = r(r-1)/t^2 + O(t^{-1})$. Part 3 of Lemma 1.5 implies the desired result.

4. The desired estimate $\nabla_{JV}c = O(\delta^{-2})$ is equivalent to $\nabla_{JV}\nu = O(1)$ because $\nabla_{JV}\nu = -c^{-2}\nabla_{JV}c$ and $c = O(\delta^{-1})$. But (27) gives that $\nabla_{JV}\nu = O(1)$.

5. The estimate $\|\nabla^2c\| = O(\delta^{-3})$ is equivalent to $\|\nabla^2\nu\| = O(\delta^{-1})$. This estimate was obtained in Step 4 above. \diamond

3. GEOMETRY OF THE CUSP: COMMONALITIES WITH SURFACES OF REVOLUTION

We continue to work locally in \mathbb{D}^* with a metric satisfying (5) and (6). In this section we establish properties of geodesics in this cuspidal region. The theme of this section is that metrics of this form inherit many of the geodesic properties of a surface of revolution for a profile function $y = x^r$, with $r > 2$. In \mathbb{R}^3 , coordinates on this surface are

$$(x, \phi) \mapsto (x, x^r \cos \phi, x^r \sin \phi), \quad x \in (0, 1], \phi \in [0, 2\pi].$$

As remarked in the introduction, if δ denotes the distance to the cusp $(0, 0, 0)$ on this surface, then $\delta(x, \phi) = x + o(x)$ and (5) holds. Other properties are:

- **Area:** The area of the region $\{\delta \leq t_0\}$ is $2\pi(r+1)^{-1}t_0^{r+1} + O(t_0^{r+2})$.
- **Clairaut Integral:** Let $\gamma(t)$ be a geodesic segment in the surface of revolution, and let $\theta(t)$ be the angle between $\dot{\gamma}(t)$ and the foliation $\{\phi = \text{const.}\}$. Then the function $t \mapsto x(\gamma(t))^r \sin \theta(t)$ is constant.

We establish here in Sections 3.1-3.3 the analogues of these properties in our setting. We also establish in Section 3.4 some coarse invariance properties of positive Jacobi fields in \mathbb{D}^* .

3.1. Area. Fix $\delta_0 > 0$, and for $t_0 \leq \delta_0$, denote by $\mathbb{D}^*(t_0)$ the disk $\delta \leq t_0$. Let c be the geodesic curvature function defined in the previous section, $d\ell$ the arclength element and $d\text{vol}$ is the area element defined by the metric.

Let $\gamma(s, t)$ be the radial unstable variation of geodesics described in the previous section, defined by the properties

- $\gamma(s, 0) = 0$, for all s ,
- $\gamma'(s, t) = V(\gamma(s, t))$, for all s, t , and
- $\partial_s \gamma(s, \delta_0) = JV(\gamma(s, \delta_0))$, for all s .

The arclength element is found by differentiating $\gamma(s, t)$ with respect to s :

$$d\ell(\gamma(s, t)) = \partial_s \gamma ds = j(s, t) ds.$$

Using part 3 of Proposition 2.1, we estimate $j(s, t)$ by

$$j(s, t) = j(s, \delta_0) \exp \left(\int_t^{\delta_0} -c(\gamma(s, x)) dx \right) = \exp \left(\int_{\delta_0}^t \left(\frac{r}{x} + O(1) \right) dx \right) = \frac{t^r}{\delta_0^r} + O(t^{r+1}),$$

and so $d\ell(\gamma(s, t)) = (t^r \delta_0^{-r} + O(t^{r+1})) ds$. We obtain that:

$$(28) \quad d\text{vol}(\gamma(s, t)) = \left(\frac{t^r}{\delta_0^r} + O(t^{r+1}) \right) ds dt.$$

The volume of the region $\mathbb{D}^*(t_0)$ is obtained by integrating $d \operatorname{vol}(\gamma(s, t))$ over the region $s \in [0, \ell(\delta_0)]$; $t \in [0, t_0]$, where $\ell(\delta_0)$ is the length of the curve $\delta = \delta_0$. It follows that

$$\operatorname{vol}(\mathbb{D}^*(t_0)) = \frac{\ell(\delta_0)t_0^{r+1}}{(r+1)\delta_0^r} + O(t_0^{r+2}).$$

3.2. The angular cuspidal functions a and b . For $v \in T^1\mathbb{D}^*$, we define $a(v), b(v)$ by:

$$(29) \quad a(v) = \langle v, V \rangle, \text{ and } b(v) = \langle v, JV \rangle.$$

Thus the vectors v with $b(v) = 0$ and $a(v) = -1$ point directly at the cusp 0 – that is, the geodesics that they determine hit the cusp in finite time – and the vectors with $b(v) = 0$ and $a(v) = 1$ point away from 0.

The functions $a, b: T^1\mathbb{D}^* \rightarrow [-1, 1]$ satisfy $a^2 + b^2 = 1$; in the example of the surface of revolution mentioned in the beginning of the section we have $a(v) = \cos \theta(v)$ and $b = \sin \theta(v)$, where $\theta(v)$ is the angle between v and the foliation $\{\phi = \text{const.}\}$, measured from the direction pointing into the cusp. Recall the definition $c(p) = \langle \nabla_{JV} V(p), JV(p) \rangle_p$ from (7). We study here how the functions a, b and c vary along a geodesic in \mathbb{D}^* .

Lemma 3.1. *Let $\gamma: [0, T] \rightarrow \mathbb{D}^*$ be a geodesic segment, and for $t \in [0, T]$, write $a(t) = a(\dot{\gamma}(t)), b(t) = b(\dot{\gamma}(t))$, and $c(t) = c(\gamma(t))$. Then:*

- (1) $\delta' = a$;
- (2) $a' = b^2 c = rb^2/\delta + O(b^2)$;
- (3) $b' = -abc = -rab/\delta + O(ab)$;

Proof. This is a straightforward verification. From the definitions, we have $\delta' = \langle \dot{\gamma}, \nabla \delta \rangle = \langle \dot{\gamma}, V \rangle = a$, and $a' = \nabla_{\dot{\gamma}} \langle \dot{\gamma}, V \rangle = \langle \dot{\gamma}, \nabla_{\dot{\gamma}} V \rangle = \langle \dot{\gamma}, bcJV \rangle = b^2 c$. Similarly, $b' = \langle \dot{\gamma}, \nabla_{\dot{\gamma}} JV \rangle = -\langle \dot{\gamma}, bcV \rangle = -abc$. We apply conclusion 3 of Proposition 2.1 to get the final estimate. \diamond

3.3. Quasi-Clairaut Relations. We next prove that there is a Clairaut-type integral for geodesic rays in \mathbb{D}^* . Recall that for $\delta_0 \in (0, 1)$, $\mathbb{D}^*(\delta_0)$ denotes the set of $z \in \mathbb{D}^*$ with $\delta(z) \leq \delta_0$.

Proposition 3.2. *If δ_0 is sufficiently small, then there exists $C = 1 + O(\delta_0)$, such that for every geodesic segment $\gamma_v: [0, T] \rightarrow \mathbb{D}^*(\delta_0)$, the following quasi-Clairaut formula holds, for all $t_1, t_2 \in [0, T]$:*

$$C^{-1} \delta(\gamma_v(t_2))^r b(\dot{\gamma}_v(t_2)) \leq \delta(\gamma_v(t_1))^r b(\dot{\gamma}_v(t_1)) \leq C \delta(\gamma_v(t_2))^r b(\dot{\gamma}_v(t_2)).$$

Proof of Proposition 3.2. First note that the statement is trivially true if $b(\dot{\gamma}_v(0)) = 0$; hence we may assume $b(\dot{\gamma}_v(t)) \neq 0$. Let $\gamma_v: [t_1, t_2] \rightarrow \mathbb{D}^*(\delta_0)$, where δ_0 will be specified later.

As in the previous section, write

$$a(t) = a(\dot{\gamma}_v(t)), b(t) = b(\dot{\gamma}_v(t)), c(t) = c(\gamma_v(t)), \text{ and } \delta(t) = \delta(\gamma_v(t)).$$

The first main ingredient in the proof of Proposition 3.2 is the following lemma.

Lemma 3.3. *For every $v \in T^1\mathbb{D}^*$, the function δ is convex along γ_v and strictly convex if $a(v) \notin \{-1, +1\}$,*

Proof. At any time t where $\delta'(t) = a(t) = 0$ we have $b(t) = 1$ so that by Lemma 3.1, we have $\delta''(t) = a'(t) = b(t)^2 c(t) = c(t)$, which is positive, by Proposition 2.1. \diamond

Returning to the proof of Proposition 3.2, let $g = \delta^r b$. We first calculate:

$$g' = (\delta^r b)' = r\delta^{r-1}b\delta' + \delta^r b' = r\delta^{r-1}ab + \delta^r \frac{-rab}{\delta} + O(|a|b\delta^r),$$

and so $g'/g = O(|a|)$. Thus there is a constant C such that $|g'/g| \leq C|a| = C|\delta'|$. Fixing $t_1 < t_2$, we have

$$\left| \int_{t_1}^{t_2} \frac{g'}{g} \right| \leq \int_{t_1}^{t_2} \left| \frac{g'}{g} \right| \leq \int_{t_1}^{t_2} C|\delta'|.$$

Thus

$$\exp \left(-C \int_{t_1}^{t_2} |\delta'| \right) \leq \frac{g(t_1)}{g(t_2)} \leq \exp \left(C \int_{t_1}^{t_2} |\delta'| \right).$$

Since δ is convex and $\delta(t_1), \delta(t_2) \leq \delta_0$, we have $\delta(t) \leq \delta_0$ for all $t \in [t_1, t_2]$, and there is at most one $t_* \in [t_1, t_2]$ where δ' vanishes. It follows easily that $\int_{t_1}^{t_2} |\delta'| \leq 2\delta_0$, which implies the conclusion. \diamond

Corollary 3.4. *Every unit-speed geodesic in \mathbb{D}^* that enters the region $\mathbb{D}^*(\delta_0)$ leaves the region in time $\leq 2\delta_0$.*

3.4. Cuspidal Jacobi fields. For γ a geodesic segment in \mathbb{D}^* , we consider solutions to the Riccati equation:

$$(30) \quad \zeta'(t) = 1 + K(\gamma(t))\zeta(t)^2,$$

which is defined on a time interval containing 0. The next lemma shows that there is a “cone condition” on initial data that is preserved by solutions to (30). We use this in the next section to construct an invariant cone field for solutions to (30) in S .

Lemma 3.5. *For every $\epsilon > 0$ there exists $\delta_0 > 0$ such that the following holds for every geodesic segment $\gamma: [0, T] \rightarrow \mathbb{D}^*(\delta_0)$. Let ζ be a solution to (30) along γ .*

- (1) *If $\zeta(0) \leq \delta(v)/(r-1-\epsilon)$, then $\zeta(t) \leq \delta(t)/(r-1-\epsilon)$ for all $t \in [0, T]$, and*
- (2) *if $\zeta(0) \geq \delta(v)/(r+\epsilon)$, then $\zeta(t) \geq \delta(t)/(r+\epsilon)$, for all $t \in [0, T]$.*

Proof. We establish the lower bound first. To this end let $w = \delta/(r+\epsilon)$. Then $w' = a/(r+\epsilon)$. To show that $w \leq \zeta$, it suffices by Lemma 1.2 to show that

$$\frac{a}{r+\epsilon} \leq 1 + Kw^2 = 1 + \left(-\frac{r(r-1)}{\delta^2} + O(\delta^{-1}) \right) \frac{\delta^2}{(r+\epsilon)^2};$$

equivalently, it suffices to show that $r - ar + 2r\epsilon - a\epsilon + \epsilon^2 \geq O(\delta)$. Since $a \leq 1$, this clearly will hold if δ_0 is sufficiently small. The upper bound is proved similarly. \diamond

4. GLOBAL PROPERTIES OF THE FLOW IN T^1S

Now consider the surface S with one puncture, satisfying the hypotheses of Theorem 1. Let δ be the distance to the cusp. For $\delta_0 > 0$, denote by $\mathcal{N}(\delta_0) = \{p : \delta(p) \leq \delta_0\}$ the convex δ_0 -neighborhood of the cusp. In this section, we modify the function δ outside of a neighborhood $\mathcal{N}(\delta_1)$ and use the modified function $\bar{\delta}$ to construct a $D\varphi_t$ -invariant cone field on TT^1S . We also use the modified function $\bar{\delta}$ to construct a new Riemannian metric on T^1S , called the \star metric, that makes T^1S complete.

Having done this, we consider the flow ψ_t on T^1S given by rescaling φ_t to have unit speed in the \star metric. We prove that this flow is Anosov in the \star metric and preserves a smooth, finite volume. This allows us to conclude that φ_t is ergodic and has smooth invariant stable and unstable foliations on which φ_t acts with bounded distortion.

4.1. Invariant cone field. In this subsection, we prove the following key technical result, which we will use to prove that a rescaled version of φ_t is Anosov.

Proposition 4.1. [Cones] *For every $\epsilon > 0$ sufficiently small, if δ_1 is sufficiently small, then the following holds.*

There exists $\beta > 0$, an extension $\bar{\delta} : S \rightarrow (0, \infty)$ of $\delta|_{\mathcal{N}(\delta_1/2)}$ and a function $\chi : S \rightarrow [\beta, r - 1 - \epsilon]$ satisfying $\chi(p) = r - 1 - \epsilon$ for $\bar{\delta}(p) \leq \delta_1$,

$$\chi(p) - \|\nabla \bar{\delta}(p)\| \geq \beta,$$

for all $p \in S$, and such that the following holds. Let $\gamma : [0, T] \rightarrow S$ be a geodesic segment in S . Then:

(1) *If u is a solution to (31) below then*

$$u(0) \in \left[\frac{\chi(\gamma(0))}{\bar{\delta}(\gamma(0))}, \frac{r + \epsilon}{\bar{\delta}(\gamma(0))} \right] \implies u(t) \in \left[\frac{\chi(\gamma(t))}{\bar{\delta}(\gamma(t))}, \frac{r + \epsilon}{\bar{\delta}(\gamma(t))} \right],$$

for all $t \in [0, T]$,

(2) *If u is a solution to (31) below then*

$$u(T) \in \left[-\frac{r + \epsilon}{\bar{\delta}(\gamma(T))}, -\frac{\chi(\gamma(T))}{\bar{\delta}(\gamma(T))} \right] \implies u(t) \in \left[-\frac{r + \epsilon}{\bar{\delta}(\gamma(t))}, -\frac{\chi(\gamma(t))}{\bar{\delta}(\gamma(t))} \right],$$

for all $t \in [0, T]$.

Proof. The proof is broken into a few steps.

4.1.1. The lower edge of the cone: $j'/j \geq g$.

Lemma 4.2. *For every $\epsilon > 0$, there exist $\mu \in (0, 1)$ and for every $\delta_0 > 0$ sufficiently small, a continuous function $g : S \rightarrow (0, \infty)$ with the following properties:*

- (1) *For every $p \in S$, we have $g(p) \leq (r - 1 - \epsilon)/\delta(p)$.*
- (2) *For every $p \in \mathcal{N}(\mu\delta_0)$, we have $g(p) = (r - 1 - \epsilon)/\delta(p)$.*
- (3) *Let $\gamma : [0, T] \rightarrow S$ be a geodesic segment in S , and let u be any solution to*

$$(31) \quad u'(t) = -K(\gamma(t)) - u(t)^2.$$

Suppose that $u(0) \geq g(\gamma(0))$. Then $u(t) \geq g(\gamma(t))$, for all $t \in [0, T]$.

Proof. Given $\epsilon < (r-2)/2$, we choose $\delta_0 \in (0, 1)$ sufficiently small according to Lemma 3.5. Let $-\kappa_0^2$ be an upper bound on the curvature on S , and let

$$\theta = \min \left\{ \kappa_0, \inf_{p \in S} \frac{r-1-\epsilon}{\delta(p)} \right\}.$$

We fix $\mu = \mu(\epsilon) > 0$ very small (to be specified later). Let $\eta: [\mu\delta_0, \delta_0] \rightarrow \mathbb{R}_{>0}$ be the affine function satisfying

$$\eta(\mu\delta_0) = r-1-\epsilon, \text{ and } \eta(\delta_0) = \theta\delta_0,$$

and define $g: S \rightarrow (0, \infty)$ by:

$$g(p) = \begin{cases} \theta & \text{if } p \in S \setminus \mathcal{N}(\delta_0), \\ \frac{\eta(\delta(p))}{\delta(p)} & \text{if } p \in \mathcal{N}(\delta_0) \setminus \mathcal{N}(\mu\delta_0), \\ \frac{r-1-\epsilon}{\delta(p)} & \text{if } p \in \mathcal{N}(\mu\delta_0). \end{cases}$$

By construction, g satisfy conditions 1 and 2. We check invariance of the condition $u(t) \geq g(\gamma(t))$; to this end, let $\gamma: [0, T] \rightarrow S$ be a geodesic, and suppose that u is a solution to (31) satisfying $u(0) \geq g(\gamma(0))$. By breaking γ into pieces if necessary, we may assume that one of the following holds:

- Case 1.** $\gamma[0, T] \subset S \setminus \mathcal{N}(\delta_0)$,
- Case 2.** $\gamma[0, T] \subset \mathcal{N}(\delta_0) \setminus \mathcal{N}(\mu\delta_0)$, or
- Case 3.** $\gamma[0, T] \subset \mathcal{N}(\mu\delta_0)$.

Cases 1 and 3 are pretty trivial. In Case 1, $g \equiv \theta$, and the fact that $-K \geq \kappa_0^2 \geq \theta^2$ implies that the condition $u \geq \theta$ is invariant. In Case 3, $g = (r-1-\epsilon)/\delta$, and we apply Lemma 3.5.

In Case 2, we will apply Lemma 1.2 to the function $u_0(t) := g(\gamma(t))$. Differentiating u_0 , we have

$$u_0'(t) = \left(\frac{\eta(\delta(\gamma(t)))}{\delta(t)} \right)' = \frac{\eta'(\delta(\gamma(t)))a(\gamma(t))}{\delta(\gamma(t))} - \frac{\eta(\delta(\gamma(t)))a(\gamma(t))}{\delta(\gamma(t))^2}.$$

Lemma 1.2 implies that $u(t) \geq u_0(t)$ for all $t \in [0, T]$ provided that $u(0) \geq u_0(0)$ and $-K(\gamma(t)) - u_0(t)^2 \geq u_0'(t)$, for all t . The latter is equivalent to:

$$(32) \quad -K(\gamma(t)) - \left(\frac{\eta(\delta(\gamma(t)))}{\delta(\gamma(t))} \right)^2 \geq \frac{\eta'(\delta(\gamma(t)))a(\gamma(t))}{\delta(\gamma(t))} - \frac{\eta(\delta(\gamma(t)))a(\gamma(t))}{\delta(\gamma(t))^2}.$$

Since $K = -r(r-1)/\delta^2 + O(1/\delta)$, if ϵ and δ_0 are sufficiently small, inequality (32) will hold provided that

$$(33) \quad r(r-1-\epsilon) - \eta^2 \geq a\eta'\delta - a\eta.$$

Since $\eta \in (0, r-1-\epsilon]$ and $\eta' < 0$, inequality (33) holds automatically when $a \geq 0$. For $a \leq 0$, inequality (33) will hold provided that for all $\delta \in [\mu\delta_0, \delta_0]$, we have:

$$(34) \quad r(r-1-\epsilon) - \eta^2 - \eta \geq -\eta'\delta.$$

Since $-\eta' \leq (r-1-\epsilon)/((1-\mu)\delta_0)$, we are reduced to proving the inequality

$$r(r-1-\epsilon) - \eta^2 - \eta \geq \frac{(r-1-\epsilon)\delta}{(1-\mu)\delta_0}.$$

To verify this, it suffices to show that the correct inequality holds at the endpoints $\delta = \mu\delta_0$ and $\delta = \delta_0$; this is easily verified provided δ_0 and $\mu = \mu(\epsilon)$ are sufficiently small. \diamond

4.1.2. *Definition of the modified distance function $\bar{\delta}$.* Fix $\epsilon < (r-2)/2$, and let $\delta_0 > 0$ and $\mu \in (0, 1)$ be given by Lemma 4.2. Let $\delta_1 = \mu\delta_0$. Since $S \setminus \mathcal{N}(\delta_1)$ is compact, we may assume that δ_0 (and hence δ_1) is small enough that

$$-\kappa_1^2(\delta_1) := \inf_{p \in S \setminus \mathcal{N}(\delta_1)} K(p) > -\frac{(r+\epsilon)(r-1+\epsilon)}{\delta_1^2}.$$

Fix $\lambda \in (0, 1)$ close enough to 1 that

$$-\kappa_1^2(\delta_1) > -\frac{(r+\epsilon)(r-1+\epsilon)}{(\lambda\delta_1)^2},$$

and $\beta_0 := \lambda(r-1-\epsilon) - 1 > 0$.

We extend δ to a C^4 function $\bar{\delta}: S \rightarrow \mathbb{R}_{>0}$ satisfying $\bar{\delta}(p) = \delta(p)$ for $p \in \mathcal{N}(\delta_1/2)$ and $\bar{\delta}(p) = \lambda\delta_1$, for $p \in S \setminus \mathcal{N}(\delta_1)$. We may do this so that $\bar{\delta}/\delta \geq \lambda$, and $\|\nabla \bar{\delta}\| \leq 1$ in $\mathcal{N}(\delta_1)$. We also denote by $\bar{\delta}$ the function on T^1S defined by $\bar{\delta}(v) = \bar{\delta}(\pi(v))$, which is constant on the fibers of T^1S . Thus if $v \in T^1S$, and $t \in \mathbb{R}$, we have:

$$\bar{\delta}(\varphi_t(v)) = \bar{\delta}(\gamma_v(t)),$$

and we will at times write these expressions interchangeably.

Let $g: S \rightarrow \mathbb{R}_{>0}$ be the lower cone function given by Lemma 4.2. Define $\chi: S \rightarrow \mathbb{R}_{>0}$ by $\chi(p) = \bar{\delta}(p)g(p)$. As with $\bar{\delta}$, we will lift χ to a function on T^1S and write $\chi(v) = \chi(\pi(v))$. Our choice of λ ensures that the following lemma holds

Lemma 4.3. *For all $p \in S$, we have $\chi(p) \leq r-1-\epsilon$. There exists $\beta > 0$ such that for all $p \in S$, we have $\chi(p) - \|\nabla \bar{\delta}(p)\| \geq \beta$.*

Proof. The first assertion follows easily from the fact that $\bar{\delta}/\delta \leq 1$ and part 1 of Lemma 4.2.

If $p \in S \setminus \mathcal{N}(\delta_1)$, then $\nabla \bar{\delta}(p) = 0$, and the conclusion holds with $\beta_1 = \inf_{S \setminus \mathcal{N}(\delta_1)} \chi > 0$. If $p \in \mathcal{N}(\delta_1)$, then $\chi(p) = (r-1-\epsilon)\bar{\delta}(p)/\delta(p)$, $\bar{\delta}(p)/\delta(p) \geq \lambda$ and $\|\nabla \bar{\delta}(p)\| \leq 1$; thus $\chi(p) - \|\nabla \bar{\delta}(p)\| \geq \lambda(r-1-\epsilon) - 1 = \beta_0 > 0$. We conclude by setting $\beta = \min\{\beta_0, \beta_1\}$. \diamond

4.1.3. *The upper edge of the cone:* $j'/j \leq (r+\epsilon)/\bar{\delta}$. Using the modified cuspidal distance function $\bar{\delta}$, we now can define an upper edge to an invariant cone field for solutions to (31).

Lemma 4.4. *Let $\bar{\delta}$ be defined as in Section 4.1.2. Let $\gamma: [0, T] \rightarrow S$ be a geodesic segment in S , and let u be any solution to (31) with $u(0) \leq (r+\epsilon)/\bar{\delta}(\gamma(0))$. Then $u(t) \leq (r+\epsilon)/\bar{\delta}(\gamma(t))$, for all $t \in [0, T]$.*

Proof. This is a straightforward application of Lemma 1.2, using only the facts that $\|\nabla \bar{\delta}\| \leq 1$, and $-K(p) \leq (r+\epsilon)(r-1+\epsilon)/\bar{\delta}(p)^2$, for all $p \in S$. \diamond

Lemmas 4.2, 4.3 and 4.4 can be applied as well to the flow φ_{-t} to obtain invariant negative cones for solutions to the equation (31). One can do this using the same functions $\chi, \bar{\delta}$ satisfying both (1) and (2) in the conclusion of Proposition 4.1. This completes the proof of the proposition. \diamond

4.2. An adapted, complete metric on T^1S . Define a new Riemannian metric on T^1S by

$$\langle (w_1, w'_1), (w_2, w'_2) \rangle_{\star, v} = \frac{1}{\bar{\delta}(v)^2} \langle w_1, w_2 \rangle_{\pi(v)} + \langle w'_1, w'_2 \rangle_{\pi(v)},$$

for $v \in T^1S$.

Remark: The \star metric on T^1S is comparable (i.e. bi-Lipschitz equivalent) to the induced Sasaki metric for the so-called Ricci metric on S . (The Ricci metric on S is obtained by scaling the original metric by $-K$.) We briefly explain.

Define a metric $\langle \cdot, \cdot \rangle_{\dagger}$ on S by conformally rescaling the original metric, as follows:

$$\langle \cdot, \cdot \rangle_{\dagger} = \bar{\delta}^{-2} \langle \cdot, \cdot \rangle.$$

This is comparable to the Ricci metric, since $-K$ is comparable to $\bar{\delta}^{-2}$.

We claim that the metric on T^1S induced by the Sasaki metric for $\langle \cdot, \cdot \rangle_{\dagger}$ is comparable to $\langle \cdot, \cdot \rangle_{\star}$. Here is a crude sketch of the proof. The unit tangent bundle T^1S for the original metric is clearly not the $\langle \cdot, \cdot \rangle_{\dagger}$ unit tangent bundle, but angles remain the same, and so $\langle \cdot, \cdot \rangle$ angular distance in the vertical fibers of T^1S coincides with $\langle \cdot, \cdot \rangle_{\dagger}$ angular distance. On the other hand, \dagger -distance in the horizontal fibers of TS (with respect to the original connection) is the original distance scaled by $\bar{\delta}^{-1}$. Thus the formulas are comparable.

As it is more convenient to work with the \star metric, we will not pursue here further the properties of the \dagger metric on S , but one can prove that (for δ_0 sufficiently small) it is complete, negatively curved with pinched curvature, and of finite volume. In the case where the original metric is the WP metric, the \dagger metric is comparable to the Teichmüller metric, which is the hyperbolic metric. We will not be using the Riemannian properties of the \star metric beyond completeness and finite volume.

Let ρ_{\star} be the Riemannian distance on T^1S induced by $\langle \cdot, \cdot \rangle_{\star}$.

Lemma 4.5. ρ_{\star} is complete.

Proof. By the Hopf-Rinow theorem, it suffices to show that any \star -geodesic is defined for all time. The only way in which a geodesic in T^1S can stop being defined is for its projection to S to hit the cusp. But the projection to S of a \star -geodesic is a curve that has speed δ when it is at distance δ from the cusp in the geometry of our Riemannian metric $\langle \cdot, \cdot \rangle$ on S . It is clear that such a curve cannot reach the cusp in finite time. \diamond

4.3. Lie brackets and \star -covariant differentiation on T^1S . If X is a vector field on S , then X has two well-defined lifts X^h and X^v to vector fields on TS , the *horizontal* and *vertical* lifts, respectively. They are defined by

$$X^h(u) = (X(\pi(u)), 0), \quad \text{and} \quad X^v(u) = (0, X(\pi(u))),$$

for $u \in T^1S$. The following formulas for Lie brackets of such lifts are standard; see [13].

Lemma 4.6. *Let X and Y be arbitrary vector fields on S . Then*

- $[X^v, Y^v]_u = 0$
- $[X^h, Y^v]_u = (0, \nabla_X Y)$
- $[X^h, Y^h]_u = ([X, Y], -R(X, Y)u)$

Recall the definitions of $\bar{V} = \nabla \bar{\delta}$ and $J\bar{V}$. To simplify notation, and since the calculations that follow are only interesting in the thin part $\mathcal{N}(\delta_1/2)$ where $\delta = \bar{\delta}$, we will write V, JV , and δ for their barred counterparts in what follows.

Lemma 4.7. *Denote by V^h, JV^h, V^v, JV^v the horizontal and vertical lifts, respectively, of V and JV . Then for $u \in \mathcal{N}(\delta_1/2)$, we have:*

- (1) $[V^v, JV^v]_u = [V^h, JV^h]_u = [V^h, V^v]_u = 0$,
- (2) $[JV^h, V^v]_u = (0, cJV)$
- (3) $[JV^h, V^h]_u = (cJV, -R(JV, V)u)$
- (4) $[JV^h, JV^v]_u = (0, -cV)$

Proof. This is a direct application of the previous lemma and the fact that $\nabla_{JV} V = [JV, V] = cJV$ from Proposition 2.1. \diamond

Observe that $\|V^h\|_\star = \|JV^h\|_\star = \delta^{-1}$, and $\|V^v\|_\star = \|JV^v\|_\star = 1$. We have:

Lemma 4.8. *Let X and Y be arbitrary vector fields on S with $\|X\| = \|Y\| = 1$, and denote by X^h, X^v, Y^h, Y^v their horizontal and vertical lifts. Then*

$$\|\nabla_{X^h}^\star Y^h\|_\star = O(\delta^{-2}), \|\nabla_{X^h}^\star Y^v\|_\star = O(\delta^{-1}), \|\nabla_{X^v}^\star Y^h\|_\star = O(\delta^{-1}),$$

and

$$\nabla_{X^v}^\star Y^v = 0.$$

In particular, the \star connection is summarized in Table 1.

	V^h	JV^h	V^v	JV^v
V^h	$-\delta^{-1}V^h$	$-\delta^{-1}JV^h$ $+\frac{1}{2}\langle R(JV, V)u, V \rangle V^v$ $+\frac{1}{2}\langle R(JV, V)u, JV \rangle JV^v$	$-\frac{\delta^2}{2}\langle R(JV, V)u, V \rangle JV^h$	$-\frac{\delta^2}{2}\langle R(JV, V)u, JV \rangle JV^h$
JV^h	$(-\delta^{-1} + c)JV^h$ $-\frac{1}{2}\langle R(JV, V)u, V \rangle V^v$ $-\frac{1}{2}\langle R(JV, V)u, JV \rangle JV^v$	$(\delta^{-1} - c)V^h$	$\frac{\delta^2}{2}\langle R(JV, V)u, V \rangle V^h$ $+cJV^v$	$\frac{\delta^2}{2}\langle R(JV, V)u, JV \rangle V^h$ $-cV^v$
V^v	$-\frac{\delta^2}{2}\langle R(JV, V)u, V \rangle JV^h$	$\frac{\delta^2}{2}\langle R(JV, V)u, V \rangle V^h$	0	0
JV^v	$-\frac{\delta^2}{2}\langle R(JV, V)u, JV \rangle JV^h$	$\frac{\delta^2}{2}\langle R(JV, V)u, JV \rangle V^h$	0	0

Table 1: $\nabla_X^\star Y(u)$, for $u \in T^1\mathcal{N}(\delta_1/2)$, where X is the row vector field, and Y is the column vector field.

Proof. The proof is a calculation using Lemma 4.7 and Koszul's formula:

$$\begin{aligned} 2\langle \nabla_X^* Y, Z \rangle_* &= X(\langle Y, Z \rangle_*) + Y(\langle X, Z \rangle_*) - Z(\langle X, Y \rangle_*) \\ &\quad + \langle [X, Y], Z \rangle_* - \langle [X, Z], Y \rangle_* - \langle [Y, Z], X \rangle_*. \end{aligned}$$

The details can be found in [8]. \diamond

Lemma 4.9. *Let $a(w) = \langle w, V(\pi(w)) \rangle, b(w) = \langle w, JV(\pi(w)) \rangle$ be defined as above. Then*

- (1) $V^h(a) = V^h(b) = 0$,
- (2) $JV^h(a) = bc$, and $JV^h(b) = -ac$,
- (3) $V^v(a) = 1$, and $V^v(b) = 0$
- (4) $JV^v(a) = 0$, and $JV^v(b) = 1$

Proof. 1. To differentiate a function ϕ on T^1S with respect to V^h at $w \in T^1S$, we parallel translate w along the geodesic $\gamma(t)$ through $\pi(w)$ tangent to V to obtain $\Pi_t^V(w)$, and then differentiate the function $\phi(\Pi_t^V(w))$ with respect to t at $t = 0$. Since γ is a geodesic tangent to V , the angle between $\Pi_t^V(w)$ and V remains constant, and so $a(\Pi_t^V(w))$ and $b(\Pi_t^V(w))$ are both constant. Thus their derivatives are both zero.

2. Let $\Pi_t^{JV}(w)$ be the parallel translate of w along the integral curve of the vector field JV through $\pi(w)$. Then

$$JV^h(a)(w) = \frac{d}{dt} \langle \Pi_t^{JV}(w), V \rangle|_{t=0} = \langle w, \nabla_{JV} V \rangle = \langle w, cJV \rangle = bc,$$

and

$$JV^h(b)(w) = \frac{d}{dt} \langle \Pi_t^{JV}(w), JV \rangle|_{t=0} = \langle w, \nabla_{JV} JV \rangle = \langle w, -cV \rangle = -ac.$$

3. To compute the derivative $V^v\phi$ at w , we differentiate $\phi(w + tV)$ at $t = 0$ in the fiber over $\pi(w)$. Thus

$$V^v(a)(w) = \frac{d}{dt} \langle w + tV, V \rangle|_{t=0} = \frac{d}{dt} t \langle V, V \rangle|_{t=0} = 1,$$

and

$$V^v(b)(w) = \frac{d}{dt} \langle w + tV, JV \rangle|_{t=0} = \frac{d}{dt} t \langle V, JV \rangle|_{t=0} = 0.$$

4. To compute the derivative $JV^v\phi$ at w , we differentiate $\phi(w + tJV)$ at $t = 0$ in the fiber over $\pi(w)$. The calculations are similar to those in 3. \diamond

Proposition 4.10. *Let X be any vector field on T^1S with $\|X\|_* = 1$. Then*

$$\|\nabla_X^* \dot{\phi}\|_* = O(\delta^{-1}).$$

In particular:

- (1) $\nabla_{V^h}^* \dot{\phi} = -a\delta^{-1}V^h - b\delta^{-1}JV^h - \frac{b^2K}{2}V^v + \frac{abK}{2}JV^v$
- (2) $\nabla_{JV^h}^* \dot{\phi} = b\delta^{-1}V^h - a\delta^{-1}JV^h + \frac{1}{2}KabV^v - \frac{1}{2}Ka^2JV^v$
- (3) $\nabla_{V^v}^* \dot{\phi} = \frac{Kab\delta^2}{2}JV^h + \left(\frac{-Kb^2\delta^2}{2} + 1 \right) V^h$
- (4) $\nabla_{JV^v}^* \dot{\phi} = \left(1 - \frac{Ka^2\delta^2}{2} \right) JV^h + \frac{Kab\delta^2}{2} V^h$

Proof. The proof is just a calculation. To see 1, for example, observe that

$$\begin{aligned}\nabla_{V^h}^* \dot{\varphi} &= \nabla_{V^h}^* (aV^h + bJV^h) = a\nabla_{V^h}^* V^h + b\nabla_{V^h}^* JV^h \\ &= -a\delta^{-1}V^h - b\delta^{-1}JV^h + \frac{b}{2}\langle R(JV, V)u, V \rangle V^v + \frac{b}{2}\langle R(JV, V)u, JV \rangle JV^v,\end{aligned}$$

where $u = aV + bJV$. Thus

$$\nabla_{V^h}^* \dot{\varphi} = -a\delta^{-1}V^h - b\delta^{-1}JV^h - \frac{b^2K}{2}V^v + \frac{abK}{2}JV^v.$$

The other formulas are proved similarly; see [8] for the details. \diamond

4.4. Time change to an Anosov flow. As above, let $\dot{\varphi}$ be the geodesic spray; i.e. the generator of the geodesic flow on T^1S . Define a new flow ψ_t on T^1S with generator

$$\dot{\psi}(v) = \bar{\delta}(v)\dot{\varphi}(v).$$

One might ask first whether this flow is complete; that is, is it defined for all time $t \in \mathbb{R}$, for each $v \in T^1S$? Note that the original flow φ_t is not complete, since it is the geodesic flow of an incomplete manifold. The completeness of ψ_t follows from the completeness of T^1S in the \star -metric defined above, and the following lemma.

Lemma 4.11. *The vector field $\dot{\psi}$ is C^3 , and there exists a constant $C > 0$ such that for every $v \in T^1S$,*

$$\|\dot{\psi}\|_{\star} = 1, \quad \text{and} \quad \|\nabla^{\star i} \dot{\psi}\|_{\star} \leq C, \quad \text{for } i = 1, 2, 3.$$

The flow $\dot{\psi}_t$ preserves a finite measure μ on $T^1\Sigma$ that is equivalent to Liouville volume for the original metric: $d\mu = \bar{\delta}^{-1}d\text{vol}$.

Proof. By definition of the \star metric, we have $\|\dot{\psi}\|_{\star} = \|\bar{\delta}\dot{\varphi}\|_{\star} = 1$.

Since $\bar{\delta}(v) = \bar{\delta}(\pi(v))$, the derivatives of $\bar{\delta}$ have no vertical component, and Corollary 2.2 gives that $\|\nabla^i \bar{\delta}\| = O(\bar{\delta}^{1-i})$. A unit horizontal vector in the \star norm is of the form $(\xi_H, 0)$, where $\|\xi_H\| = \bar{\delta}$. Thus

$$(35) \quad \|\nabla^{\star i} \bar{\delta}\|_{\star} = \bar{\delta}^i \|\nabla^i \bar{\delta}\| = O(\bar{\delta}),$$

for $i = 1, 2, 3$.

Proposition 4.10 implies $\|\nabla^{\star} \dot{\varphi}\|_{\star}$ has magnitude $\bar{\delta}(v)^{-1}$ in the \star metric. A similar calculation taking higher covariant derivatives of the formulas in Proposition 4.10 and using the facts that $\|\nabla K\| = O(\bar{\delta}^{-3})$, $\|\nabla^2 K\| = O(\bar{\delta}^{-4})$, $\|\nabla^i c\| = \delta^{-1-i}$, and (35) gives that

$$(36) \quad \|\nabla^{\star i} \dot{\varphi}\|_{\star} = O(\bar{\delta}^{-1}),$$

for $i = 1, 2, 3$.

Combining (36) and (35), we obtain that

$$\|\nabla^{\star}(\bar{\delta}\dot{\varphi})\|_{\star} = \|\nabla^{\star}(\bar{\delta})\dot{\varphi}\|_{\star} + \|\bar{\delta}\nabla^{\star}\dot{\varphi}\|_{\star} \leq \|\nabla^{\star}(\bar{\delta})\|_{\star}\|\dot{\varphi}\|_{\star} + \bar{\delta}\|\nabla^{\star}\dot{\varphi}\|_{\star} = O(1).$$

Similarly, we obtain that $\|\nabla^{\star i}(\bar{\delta}\dot{\varphi})\|_{\star} = O(1)$, for $i = 2, 3$.

Let ω be the canonical one form on the tangent bundle TS with respect to the original metric. Then $\varphi_t^* \omega = \omega$, for all t , and $d\text{vol} = \omega \wedge d\omega$ on T^1S . We have that:

$$\mathcal{L}_{\dot{\psi}}(\bar{\delta}^{-1}\omega \wedge d\omega) = d\left(\iota_{\dot{\psi}}(\bar{\delta}^{-1}\omega \wedge d\omega)\right) = d(d\omega) = 0,$$

since $\bar{\delta}^{-1}\omega(\dot{\psi}) = \omega(\dot{\varphi}) \equiv 1$. Thus ψ_t preserves the smooth measure μ defined by $d\mu = \bar{\delta}^{-1}\omega \wedge d\omega = \bar{\delta}^{-1}d\text{vol}$.

To see that $\mu(T^1S) < \infty$, we use the expression for $d\text{vol}$ from (28) and integrate:

$$\mu(T^1S) = \int_{T^1S} \bar{\delta}^{-1}d\text{vol} = O\left(\int_0^{\delta_0} x^{r-1}dx\right) < \infty.$$

◇

The flow ψ_t is a time change of φ_t ; that is, it has the same orbits, but they are traversed at a different speed, depending on the distance to the singular locus. Indeed, defining the cocycle $\tau: T^1S \times \mathbb{R} \rightarrow \mathbb{R}$ by the implicit formula

$$(37) \quad \int_0^{\tau(v,t)} \frac{dx}{\bar{\delta}(\varphi_x(v))} = t,$$

we have that $\psi_t(v) = \varphi_{\tau(v,t)}(v)$, for all $v \in T^1S$, $t \in \mathbb{R}$. This gives an alternate way to see the completeness of the flow ψ : the function $\bar{\delta}$ clearly remains positive along orbits of ψ for all time.

Theorem 4.12. *The flow ψ_t is an Anosov flow in the \star -metric. That is, there exists a $D\psi_t$ -invariant, continuous splitting of the tangent bundle:*

$$T(T^1S) = E_\psi^u \oplus \mathbb{R}\dot{\psi} \oplus E_\psi^s$$

and constants $C > 0$, $\lambda > 1$ such that for every $v \in T^1S$, and every $t > 0$:

- $\xi \in E_\psi^u(v) \implies \|D\psi_{-t}(\xi)\|_\star \leq C\lambda^{-t}\|\xi\|_\star$, and
- $\xi \in E_\psi^s(v) \implies \|D\psi_t(\xi)\|_\star \leq C\lambda^{-t}\|\xi\|_\star$.

From Theorem 4.12 we obtain several important properties of both ψ_t and φ_t . The first is ergodicity. Since volume preserving Anosov flows are ergodic, the flows φ_t and ψ_t have the same orbits, and the \star volume is equivalent to (i.e. has the same zero sets as) the original volume on T^1S , we obtain:

Corollary 4.13. *The flow ψ_t is ergodic with respect to the invariant volume μ . Consequently, φ_t is ergodic with respect to volume.*

In the next corollary we obtain a splitting of TT^1S , invariant under $D\varphi_t$.

Corollary 4.14. *$D\varphi_t$ has an invariant singular hyperbolic splitting*

$$T(T^1S) = E_\varphi^u \oplus \mathbb{R}\dot{\varphi} \oplus E_\varphi^s,$$

with E_φ^u and E_φ^s given by intersecting $E_\psi^u \oplus \mathbb{R}\dot{\psi}$ and $E_\psi^s \oplus \mathbb{R}\dot{\psi}$ with the smooth, $D\varphi_t$ -invariant bundle $\dot{\varphi}^\perp$.

Since the weak stable and unstable distributions of a C^3 Anosov flow in dimension 3 are $C^{1+\alpha}$, for some $\alpha > 0$, we also obtain:

Corollary 4.15. *The distributions $E_\psi^u \oplus \mathbb{R}\dot{\psi}$ and $E_\psi^s \oplus \mathbb{R}\dot{\psi}$ are $C^{1+\alpha}$, for some $\alpha > 0$. The distributions E_φ^u and E_φ^s are also $C^{1+\alpha}$, when measured in the \star metric. Thus in the compact part $\bar{\delta} \geq \delta_0$, the distributions E_φ^u and E_φ^s are uniformly $C^{1+\alpha}$.*

Remark: If $\bar{\delta}(p) \leq \delta_1/2$, then the vector $V(p) = \nabla \bar{\delta}(p)$ points directly away from the cusp. It is not difficult to see that the unstable manifold $\mathcal{W}_\psi^u(V(p))$ consists of the restriction of the vector field V to the circle $\bar{\delta} = \bar{\delta}(v)$. For these vectors, the unstable bundles E_ψ^u and E_φ^u coincide.

Finally we obtain the key bounds on distortion for the flow φ_t that will be used to prove exponential mixing.

Corollary 4.16 (Distortion control). *For $t \in \mathbb{R}$ and $v \in T^1S$ denote by $\|D_v^s \psi_t\|_\star$ and $\|D_v^u \psi_t\|_\star$ the \star -norm of the restriction of $D_v \psi_t$ to E_ψ^s and E_ψ^u , respectively. Similarly define $\|D_v^s \varphi_t\|_\star$ and $\|D_v^u \varphi_t\|_\star$ using the bundles E_φ^s and E_φ^u . There exist $\theta > 0$, $C \geq 1$ and $\sigma > 0$ such that for every $v \in T^1S$:*

- (1) *If $w \in \mathcal{W}_\psi^s(v, \sigma)$ and $w' \in \mathcal{W}_\psi^u(v, \sigma)$, then for all $t > 0$:*

$$|\log \|D_v^s \psi_t\|_\star - \log \|D_{w'}^s \psi_t\|_\star| \leq C \rho_\star(v, w)^\theta,$$

and

$$|\log \|D_v^u \psi_{-t}\|_\star - \log \|D_{w'}^u \psi_{-t}\|_\star| \leq C \rho_\star(v, w')^\theta.$$

- (2) *If $w \in \mathcal{W}_\varphi^s(v, \sigma)$ and $w' \in \mathcal{W}_\varphi^u(v, \sigma)$, then for all $t > 0$:*

$$|\log \|D_v^s \varphi_t\|_\star - \log \|D_{w'}^s \varphi_t\|_\star| \leq C \rho_\star(v, w)^\theta,$$

and

$$|\log \|D_v^u \varphi_{-t}\|_\star - \log \|D_{w'}^u \varphi_{-t}\|_\star| \leq C \rho_\star(v, w')^\theta.$$

Proof. The results for ψ_t are standard properties of Anosov flows. For φ_t , we need only note that the map induced by φ_t between any two \mathcal{W}_φ^s manifolds on the same orbit is just the composition of the map induced by ψ_t between the corresponding \mathcal{W}_ψ^s manifolds with projections along flow lines at both ends between the \mathcal{W}_φ^s and \mathcal{W}_ψ^s manifolds. These latter projections are uniformly C^2 . \diamond

Remark: It is not hard to see that the stable and unstable bundles E_ψ^u and E_ψ^s are not jointly integrable. It follows that the Anosov flow ψ_t is mixing with respect to the measure μ . This leads to the question: is ψ_t exponentially mixing (if, for example, δ_0 is chosen small enough in the construction)?

4.5. Proof of Theorem 4.12. By standard arguments in smooth dynamics, to prove that ψ_t is an Anosov flow, it suffices to find nontrivial cone fields \mathcal{C}^+ and \mathcal{C}^- over T^1S and constants $C > 0$, $\lambda > 1$ with the properties:

- $\mathcal{C}^+(v) \cap \mathcal{C}^-(v) = \{0\}$, and $\mathcal{C}^\pm(v) \cap \mathbb{R}\dot{\psi} = \{0\}$;
- $D_v \psi_1(\mathcal{C}^+(v)) \subset \mathcal{C}^+(\psi_1(v))$, and $D_v \psi_{-1}(\mathcal{C}^-(v)) \subset \mathcal{C}^-(\psi_{-1}(v))$; and
- For all $t > 0$, and all $\xi^+ \in \mathcal{C}^+(v)$ and $\xi^- \in \mathcal{C}^-(v)$, we have

$$\|D_v \psi_t(\xi^+)\|_\star \geq C \lambda^t \text{ and } \|D_v \psi_{-t}(\xi^-)\|_\star \geq C \lambda^t.$$

The derivative of ψ_t restricted to $\dot{\varphi}^\perp$ has a component in the $\dot{\varphi}$ direction of TT^1S owing to the time change. We have:

$$(38) \quad D_v \psi_t(\xi) = D_v \varphi_{\tau(v,t)}(\xi) = D_v \varphi_s|_{s=\tau(v,t)}(\xi) + D_v \tau(v,t)(\xi) \dot{\varphi}(v).$$

Our strategy to find the cone fields is summarized in two steps.

- (1) Use the properties of $D_v\varphi_{\tau_t}$ previously obtained in Lemma 4.2 to define the perpendicular components (i.e. in $\dot{\varphi}^\perp$) of \mathcal{C}^\pm .
- (2) Using a bound on the “shear term” $D_v\tau_t(\xi)$ in the \star norm, we then define the components of \mathcal{C}^\pm in the $\mathbb{R}\dot{\varphi}$ direction.

We carry out these steps in the following sections.

4.5.1. Action of $D\varphi_s$. Here we fix $t > 0$ and study the action of $D_v\varphi_s: \dot{\varphi}^\perp(v) \rightarrow \dot{\varphi}^\perp(\varphi_s(v))$ at $s = \tau(v, t)$. The derivatives of τ do not enter into these calculations; we are essentially establishing properties of the original flow φ_s (as measured in the \star -metric).

Proposition 4.17. *For any $v \in T^1S$, and any real numbers y_0, z_0 satisfying $z_0/y_0 \in [\chi(v), r + \epsilon]$, the following holds. For $s \geq 0$, define y_s and z_s by:*

$$D_v\varphi_s(y_0\bar{\delta}(v)Jv, z_0Jv) = (y_s\bar{\delta}(\gamma_v(s))J\dot{\gamma}_v(s), z_sJ\dot{\gamma}_v(s)).$$

Then:

- (1) $z_s/y_s \in [\chi(\varphi_s(v)), r + \epsilon]$, for all $s \geq 0$, and
- (2) for every $t > 0$:

$$\frac{y_{\tau(v,t)}}{y_0} \geq e^{\beta t}.$$

Proof. For $s > 0$, let $j(s) = \bar{\delta}(\gamma_v(s))y_s$. Then, since $(j(s)J\dot{\gamma}_v(s), j'(s)J\dot{\gamma}_v(s))$ is a perpendicular Jacobi field, the definition of y_s, z_s implies that $j'(s) = z_s$. In particular, $j'(s)/j(s) = z_s/\bar{\delta}(\gamma_v(s))y_s$.

Suppose that $z_0/y_0 \in [\chi(v), r + \epsilon]$. Then

$$(39) \quad j'(s)/j(s) \in \left[\frac{\chi(\gamma_v(s))}{\bar{\delta}(\gamma_v(s))}, \frac{r + \epsilon}{\bar{\delta}(\gamma_v(s))} \right]$$

holds for $s = 0$, and Proposition 4.1 implies that (39) holds for all $s > 0$. We conclude that $z_s/y_s \in [\chi(\varphi_s(v)), r + \epsilon]$, for all $s > 0$.

Turning to the second item in the proposition, we have that

$$\frac{\bar{\delta}(\gamma_v(s))y_s}{\bar{\delta}(\gamma_v(0))y_0} = \frac{j(s)}{j(0)} = \exp\left(\int_0^s \frac{j'(u)}{j(u)} du\right),$$

which gives that $y_s/y_0 = \bar{\delta}(\gamma_v(0))/\bar{\delta}(\gamma_v(s)) \exp(\int_0^s z_u/(\bar{\delta}(\gamma_v(u))y_u) ds)$, and so

$$y_s = y_0 \exp\left(\int_0^s \frac{-D\bar{\delta}(\dot{\varphi}_u(v))}{\bar{\delta}(\varphi_u(v))} + \frac{z_u}{y_u\bar{\delta}(\varphi_u(v))} du\right).$$

Since $z_u/y_u \geq \chi(\varphi_u(v))$, for $u \leq s$, and $\chi - \|\nabla\bar{\delta}\| > \beta$, we have

$$\frac{y_s}{y_0} \geq \exp\left(\int_0^s \frac{\beta}{\bar{\delta}(\varphi_u(v))} du\right).$$

We make the substitution $s = \tau(v, t)$ and use the fact that $\int_0^{\tau(v,t)} \bar{\delta}(\varphi_u(v))^{-1} du = t$ to obtain the conclusion. \diamond

4.5.2. Invariant cone fields. We define invariant stable and unstable cones \mathcal{C}^- and \mathcal{C}^+ . The angle between \mathcal{C}^+ and \mathcal{C}^- will be uniformly bounded in the \star -metric, as will be the angle between either of them and $\dot{\psi}$. We establish the properties of \mathcal{C}^+ in detail; the analogous properties for \mathcal{C}^- are obtained by the same proof, reversing the direction of time.

Fix $B > 0$ to be specified later, and let

$$\mathcal{C}^+(v) := \{(\bar{\delta}(v)(xv + yJv), zJv) : z/y \in [\chi(v), r + \epsilon] \text{ \& } |x| \leq B|y|\} \cup \{0\},$$

and

$$\mathcal{C}^-(v) := \{(\bar{\delta}(v)(xv + yJv), zJv) : z/y \in [-(r + \epsilon), -\chi(v)] \text{ \& } |x| \leq B|y|\} \cup \{0\}.$$

Note that since χ is bounded below away from 0, if $\xi = (\bar{\delta}(v)(xv + yJv), zJv) \in \mathcal{C}^\pm(v)$, then $\|\xi\|_\star$ is uniformly comparable to both $|y|$ and $|z|$.

Lemma 4.18. *If $B > 0$ is sufficiently large, then $D_v\psi_1(\mathcal{C}^+(v)) \subset \mathcal{C}^+(\psi_1(v))$, and there exists $C > 0$ such that*

$$\xi \in \mathcal{C}^+(v) \implies \|D\psi_t(\xi)\|_\star \geq Ce^{\beta t}\|\xi\|_\star,$$

for all $t \geq 0$.

Proof. Recall that $\dot{\psi}(v) = (\bar{\delta}(v)v, 0)$, and $D_v\psi_t(\dot{\psi}(v)) = \dot{\psi}(\psi_t(v))$. Applying $D_v\psi_t$ to $\xi = (\bar{\delta}(v)(x_0v + y_0Jv), z_0Jv)$ and using (38), we get

$$\begin{aligned} D_v\psi_t(\xi) &= D_v\psi_t(x_0\dot{\psi}(v)) + D_v\psi_t(\bar{\delta}(v)y_0Jv, z_0Jv) \\ &= D_v\psi_t(x_0\dot{\psi}(v)) + D_v\varphi_s|_{s=\tau(v,t)}(y_0\bar{\delta}(v)Jv, z_0Jv) + D_v\tau(v,t)(y_0\bar{\delta}(v)Jv, z_0Jv)\dot{\varphi}(\psi_t(v)) \\ &= (x_0\bar{\delta}(\psi_t(v)) + D_v\tau(v,t)(y_0\bar{\delta}(v)Jv, z_0Jv))(\psi_t(v), 0) \\ &\quad + D_v\varphi_s|_{s=\tau(v,t)}(y_0\bar{\delta}(v)J\psi_t(v), z_0J\psi_t(v)). \\ &=: (x_\tau\bar{\delta}(\varphi_\tau(v))\varphi_\tau(v), 0) + (y_\tau\bar{\delta}(\varphi_\tau(v))J\varphi_\tau(v), z_\tau J\varphi_\tau(v)), \end{aligned}$$

where in the last expression we've used the abbreviation $\tau = \tau(v, t)$.

Assume that $\xi \in \mathcal{C}^+(v)$ and without loss of generality that $y_0 \geq 0$. This implies that $z_0 \in [\chi(v)y_0, (r + \epsilon)y_0]$ and $|x_0| \leq B|y_0|$. Proposition 4.17 implies that for any $t > 0$:

$$\begin{aligned} |x_\tau| &= |x_0 + \frac{1}{\bar{\delta}(\varphi_\tau(v))}D_v\tau(v,t)(y_0\bar{\delta}(v)Jv, z_0Jv)| \leq |x_0| + \|D_v\tau(t, \cdot)\|_\star \|(y_0\bar{\delta}(v)Jv, z_0Jv)\|_\star \\ &\leq |x_0| + (1 + 1/\beta)\|D_v\tau(t, \cdot)\|_\star|z_0|, \end{aligned}$$

$y_\tau \geq e^{\beta t}y_0$, and $z_\tau \in [\chi(\varphi_\tau(v))y_\tau, (r + \epsilon)y_\tau]$.

Now fix $t = 1$, and let $\tau_1 = \tau(v, 1)$. We want to show that $D_v\psi_1(\xi) \in \mathcal{C}^+(\psi_1(v)) = \mathcal{C}^+(\varphi_{\tau_1}(v))$; i.e. that $z_{\tau_1}/y_{\tau_1} \in [\chi(\psi_1(v)), r + \epsilon]$ and $|x_{\tau_1}| \leq B|y_{\tau_1}|$.

From the previous discussion, we have $z_{\tau_1} \in [\chi(\varphi_{\tau_1}(v))y_{\tau_1}, (r + \epsilon)y_{\tau_1}]$, and setting $C_1 = (1 + 1/\beta)\|D_v\tau(1, \cdot)\|_\star$, we also have:

$$|x_{\tau_1}| \leq |x_0| + C_1|z_0| \leq By_0 + C_1|z_0| \leq (B + C_1(r + \epsilon))y_0 \leq (B + C_1e(r + \epsilon))e^{-\beta}y_{\tau_1}.$$

Thus we want choose B such that $(B + C_1(r + \epsilon))e^{-\beta} \leq B$, which holds if

$$B \geq \frac{C_1(r + \epsilon)e^{-\beta}}{1 - e^{-\beta}}.$$

A similar argument works for $y_0 < 0$.

Finally if $\xi \in \mathcal{C}^+(v)$, then $\|D\psi_t(\xi)\|_\star$ is uniformly comparable to $|y_\tau|$; since $|y_\tau|$ grows exponentially on the order $e^{\beta t}$, so does $\|D_v\psi_t\xi\|_\star$. \diamond

5. EXPONENTIAL MIXING

As mentioned in the introduction, to prove exponential mixing of φ_t , we will construct a Young tower – a special section to the flow – whose return times have exponential tails. Since orbits of φ_t spend only a bounded amount of time in the cuspidal region, ensuring exponential tails for the return time is not difficult.

Fix $\delta_2 \leq \delta_1/2$ sufficiently small, and denote by $Z = Z(\delta_2)$ the circle $\{p : \bar{\delta}(p) = \delta_2\}$. Then Z lifts to two distinguished circles $Z^u, Z^s \subset T_Z^1 S$ in the unit tangent bundle:

$$Z^u := \{\nabla \bar{\delta}(p) : p \in Z\}, \text{ and } Z^s := \{-\nabla \bar{\delta}(p) : p \in Z\}.$$

Then Z^u is a closed leaf of the unstable foliation \mathcal{W}^u for φ_t , and Z^s is a closed leaf of the stable foliation. In a neighborhood U of Z^u in $T^1 S$, there is a well-defined projection $\pi^{cs} : U \rightarrow Z^u$ along local leaves of the weak-stable foliation \mathcal{W}^{cs} for φ_t . Since the foliation \mathcal{W}^{cs} is $C^{1+\alpha}$, the map π^{cs} is a $C^{1+\alpha}$ fibration.

We will prove:

Theorem 5.1. *For any $v_0 \in Z^u$, there are constants $C \geq 1$, $\lambda, \alpha \in (0, 1)$, a collection of disjoint, open subintervals $\mathcal{I} = \{\Delta_j : j \geq 1\}$, with $\Delta_j \subset \Delta_0 := Z^u \setminus \{v_0\}$, for $j \geq 1$, and a function $R : \bigcup \mathcal{I} \rightarrow [C^{-1}, \infty)$ such that:*

- (1) $|\Delta_0 \setminus \bigcup \mathcal{I}| = 0$, where $|\cdot|$ denotes Lebesgue measure on unstable leaves.
- (2) For each $v \in \bigcup \mathcal{I}$, there exists $v' \in \Delta_0$ such that $\varphi_{R(v)}(v) \in \mathcal{W}_{loc}^s(v')$.
- (3) Define $F : \mathcal{I} \rightarrow \Delta_0$ by $F(v) = \pi^{cs} \varphi_{R(v)}(v)$. For each $j \geq 1$ there is a diffeomorphism $h_j : \Delta_0 \rightarrow \Delta_j$ such that for all $v \in \Delta_0$:

$$F \circ h_j(v) = v.$$

- (4) h_j is a uniform contraction: $d(h_j(v_1), h_j(v_2)) \leq \lambda$.
- (5) $\log h'_j$ is uniformly $C^{1+\alpha}$:

$$|\log h'_j(v_1) - \log h'_j(v_2)| \leq C d(v_1, v_2)^\alpha,$$

for all $v_1, v_2 \in \Delta_0$.

- (6) $\|(R \circ h_j)'\|_\infty \leq C$ for all j .
- (7) For each $k > 0$, we have $|\{v \in \bigcup \mathcal{I} : R(v) \geq k\}| \leq C \lambda^k$; moreover, there exists $\epsilon > 0$ such that

$$\sum_j \exp(\epsilon |R \circ h_j|_\infty) |h'_j|_\infty < \infty.$$

- (8) (UNI Condition) For $n \geq 1$, let $R_n = \sum_{i=0}^{n-1} R \circ F^i$ (where defined) and let

$$\mathcal{H}_n = \{h_{\mathbf{j}} := h_{j_n} \circ h_{j_{n-1}} \circ \cdots \circ h_{j_1} : \mathbf{j} = (j_1, \dots, j_n), j_k \geq 1\}$$

be the set of inverse branches of F^n , which satisfy $F^n \circ h_{\mathbf{j}} = \text{id}_{\Delta_0}$, for all $h_{\mathbf{j}} \in \mathcal{H}_n$. Then there exists $D > 0$ such that, for all $N \geq 1$, there exist $n \geq N$ and $h_{\mathbf{j}_1}, h_{\mathbf{j}_2} \in \mathcal{H}_n$ such that

$$\inf_{v \in \Delta_0} |(R_n \circ h_{\mathbf{j}_1} - R_n \circ h_{\mathbf{j}_2})'(v)| \geq D.$$

A recent result of Araújo-Melbourne [4] shows that conditions (1)–(8) imply exponential mixing of φ_t . For $\theta \in (0, 1]$, define $C^\theta(T^1S)$ to be the set of L^∞ functions $u: T^1S \rightarrow \mathbb{R}$ such that $\|u\|_\theta := |u|_\infty + |u|_\theta < \infty$, where

$$|u|_\theta := \sup_{v \neq v'} \frac{|u(v) - u(v')|}{\rho(v, v')^\theta}.$$

Corollary 5.2. *The flow φ_t is exponentially mixing: for every $\theta \in (0, 1]$, there exist constants $c, C > 0$ such that for every $u_1, u_2 \in C^\theta(T^1S)$, we have*

$$\left| \int_{T^1S} u_1 u_2 \circ \varphi_t d\text{vol} - \int u_1 d\text{vol} \int u_2 d\text{vol} \right| \leq C e^{-ct} \|u_1\|_\theta \|u_2\|_\theta,$$

for all $t > 0$.

Proof. In the language of [4], conditions (1)–(8) in Theorem 5.1 imply that we can express the ergodic flow φ_t as the natural extension of a $C^{1+\alpha}$ skew product flow satisfying the UNI condition. See the discussion in [4] after Remark 4.1. Theorem 3.3 in [4] then applies to give that φ_t is exponentially mixing for a suitable function space of observables, in particular those that are C^3 . A standard mollification argument gives exponential mixing for observables in C^θ (see Remark 3.4 in [4]). \diamond

The construction is carried out in two parts. First, in Subsection 5.1, we isolate those orbits that leave the thick part of T^1S and travel deeply into the cusp. These orbits are easily described on a topological level using the Quasi-Clairaut relation developed in Section 3.3. We give a precise description of the first return map to the thick part for these orbits. Next, in Subsection 5.2, we analyze the orbits beginning in Z^u and decompose into pieces visiting the thin part in a controlled way. We combine these analyses to obtain the desired decomposition of Δ_0 in Theorem 5.1.

5.1. Constructing sections to the flow in the cusp. We will work with $\delta_2 \leq \delta_1/2$ so that for $\bar{\delta}(p) \leq \delta_2$, we have $\bar{\delta}(p) = \delta(p)$ and $\chi(p) = r - 1 - \epsilon$, where χ is the function appearing in Proposition 4.1.

Let $\mathcal{T}(\delta_2) \subset T^1S$ be the torus consisting of all unit tangent vectors to S with footpoint in $Z(\delta_2)$:

$$\mathcal{T}(\delta_2) = T_{Z(\delta_2)}^1 S.$$

This torus is transverse to the vector field $\dot{\varphi}$, except at the two circles

$$C^+ := \{J\nabla\bar{\delta}(p) : p \in Z\}, \text{ and } C^- := \{-J\nabla\bar{\delta}(p) : p \in Z\}.$$

Let $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^s$ be the laminations of $\mathcal{T}(\delta_2)$ obtained by intersecting leaves of the weak foliations \mathcal{W}^{cu} and \mathcal{W}^{cs} with $\mathcal{T}(\delta_2)$. On $\mathcal{T}(\delta_2) \setminus (C^+ \cup C^-)$, the laminations $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^s$ are transverse foliations with 1-dimensional leaves. Each lamination $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^s$ has exactly one closed leaf, the curves Z^u and Z^s respectively, which are also unstable and stable manifolds for φ_t .

For $\eta_0 > 0$ we define two open subsets $\mathcal{T}_{in}(\delta_2, \eta_0)$ and $\mathcal{T}_{out}(\delta_2, \eta_0)$ of $\mathcal{T}(\delta_2)$ as follows:

$$\mathcal{T}_{in}(\delta_2, \eta_0) := \{v \in \mathcal{T}(\delta_2) : a(v) < 0 \text{ \& } |b(v)| \leq \eta_0\},$$

and

$$\mathcal{T}_{out}(\delta_2, \eta_0) := \{v \in \mathcal{T}(\delta_2) : a(v) > 0 \text{ \& } |b(v)| \leq \eta_0\},$$

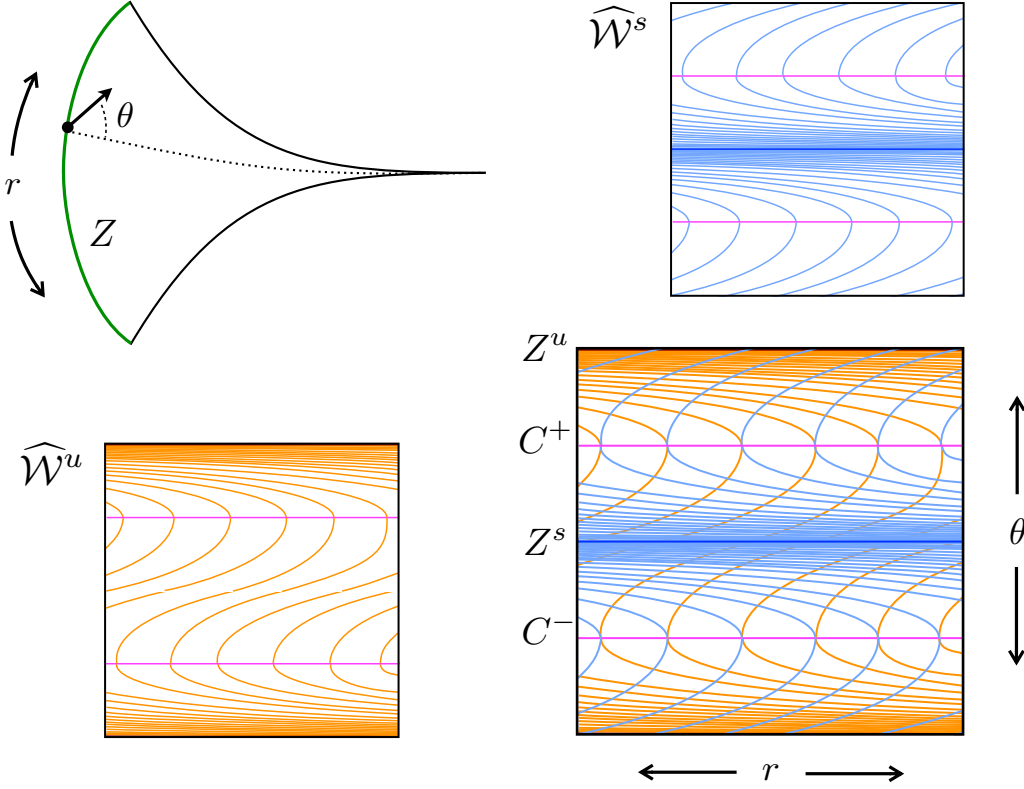


FIGURE 1. The laminations $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^s$. The singular loci C^\pm are the labeled circles where the leaves of $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^s$ become tangent.

where a and b are defined by (29). Note that $Z^u \subset \mathcal{T}_{out}(\delta_2, \eta_0)$, and $Z^s \subset \mathcal{T}_{in}(\delta_2, \eta_0)$, for all $\eta_0 > 0$.

If $\eta_0 < 1$, then $\mathcal{T}_{out}(\delta_2, \eta_0)$ and $\mathcal{T}_{in}(\delta_2, \eta_0)$ are disjoint from C^\pm , and so $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^s$ form uniformly transverse foliations in these cylinders.

Proposition 3.2 implies that if δ_2 is sufficiently small, then for all $\eta_0 < 1/2$, there is a well-defined first return map

$$\mathcal{R}: \mathcal{T}_{in}(\delta_2, \eta_0) \setminus Z^s \rightarrow \mathcal{T}_{out}(\delta_2, 2\eta_0)$$

for the flow φ_t , with a local inverse $\mathcal{R}^{-1}: \mathcal{T}_{out}(\delta_2, \eta_0) \setminus Z^u \rightarrow \mathcal{T}_{in}(\delta_2, 2\eta_0)$. These maps, where defined, are C^3 and preserve the foliations $\widehat{\mathcal{W}}^u$ and $\widehat{\mathcal{W}}^s$.

Fix $v_0 \in Z^u$ and recall that $\mathcal{W}^u(v_0) = \widehat{\mathcal{W}}^u(v_0) = Z^u$. Fix η small, and let $I_0 = \widehat{\mathcal{W}}^s(v_0, \eta)$. The image of $I_0 \setminus \{v_0\}$ under \mathcal{R}^{-1} is the union of two infinite rays spiraling into the unique closed stable manifold Z^s in $\mathcal{T}(\delta_2)$. Fix another point $v'_0 \in Z^s$ (for example, $v'_0 = -v_0$), and fix two points $v'_L, v'_R \in \widehat{\mathcal{W}}^u(v'_0, \eta) \cap \mathcal{R}^{-1}(I_0)$, to the left and right, respectively, of v'_0 in $\widehat{\mathcal{W}}^u(v'_0, \eta)$ with respect to some fixed orientation.

Let $v_L = \mathcal{R}(v'_L)$, and let $v_R = \mathcal{R}(v'_R)$. The unstable manifold $\widehat{\mathcal{W}}^u(v_L)$ contains an infinite ray from v_L , spiraling into Z^u from the left and cutting I_0 infinitely many times. Let w_L be the first intersection point of this ray with I_0 ; it lies to the right of v_L , and

to the left of v_0 . The points v_L, w_L define a closed curve c_L in $\mathcal{T}(\delta_2)$, consisting of the piece of $\widehat{\mathcal{W}}^u(v_L)$ connecting v_L to w_L and the subinterval of $I_0 = \widehat{\mathcal{W}}^s(v_0, \eta)$ from v_L to w_L .

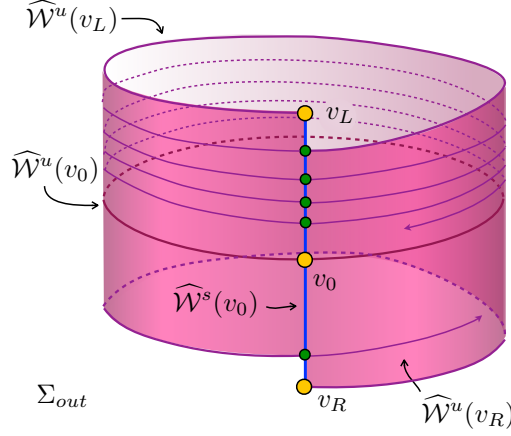


FIGURE 2. The section Σ_{out} of vectors pointing out of the cusp.

Similarly, let w_R be the first intersection of the infinite ray $\widehat{\mathcal{W}}^u(v_R)$ spiraling into Z^u from the right, and let c_R be the curve constructed analogously. The two curves c_L and c_R bound a cylindrical region Σ_{out} in $\mathcal{T}(\delta_2)$, which is depicted in Figure 2.

Let $\pi^s: \Sigma_{out} \rightarrow Z^u$ be the projection along leaves of $\widehat{\mathcal{W}}^s$, which is simply the restriction of the projection π^{cs} previously defined to the domain Σ_{out} . Then π^s is $C^{1+\alpha}$ and maps the boundary curves c_L and c_R onto Z^u . The map π^s is a diffeomorphism when restricted to the interior of any interval of $\widehat{\mathcal{W}}^s$ that begins and ends in I_0 and makes one revolution around Σ_{out} .

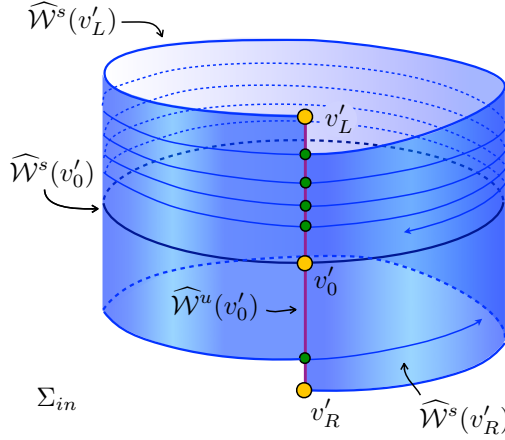


FIGURE 3. The section Σ_{in} of vectors pointing toward the cusp.

We define the section $\Sigma_{in} \subset \mathcal{T}_{in}(\delta_2, \eta_0)$ similarly: it is bounded by two curves c'_L and c'_R , where c'_L is the union of two segments of $\widehat{\mathcal{W}}^u(v'_L)$ and $\widehat{\mathcal{W}}^s(v'_L)$, and c'_R is the

union of segments of $\widehat{\mathcal{W}}^u(v'_R)$ and $\widehat{\mathcal{W}}^s(v'_R)$. See Figure 3. By construction, we have that $\mathcal{R}(c'_L) = c_L$, $\mathcal{R}(c'_R) = c_R$, and:

$$\mathcal{R}(\Sigma_{in} \setminus Z^s) = \Sigma_{out} \setminus Z^u.$$

If the radius η of I_0 was initially chosen sufficiently small, then there exists $\eta_0 \in (0, 1/8)$ such that

$$(40) \quad \mathcal{T}_{out}(\delta_2, \eta_0/2) \subset \Sigma_{out} \subset \mathcal{T}_{out}(\delta_2, \eta_0), \text{ and } \mathcal{T}_{in}(\delta_2, \eta_0/2) \subset \Sigma_{in} \subset \mathcal{T}_{in}(\delta_2, \eta_0).$$

Fix this η_0 .

Let $\pi^u: \Sigma_{in} \rightarrow Z^s$ be the projection along leaves of $\widehat{\mathcal{W}}^u$, which is the restriction of the center-unstable π^{cu} to Σ_{in} . The fibers of π^u are pieces of $\widehat{\mathcal{W}}^u$ -unstable manifold.

Let us examine the return time function for the flow on the fibers of π^u . Let \mathcal{N} be a small neighborhood of $\Sigma_{in} \setminus Z^s$ defined by flowing $\Sigma_{in} \setminus Z^s$ under φ_t in a small time interval. For $v \in \mathcal{N}$, let $t_{\mathcal{R}}(v)$ be the smallest time $t > 0$ satisfying $\varphi_t(v) \in \Sigma_{out}$:

$$(41) \quad t_{\mathcal{R}}(v) = \inf \{t > 0 : \varphi_t(v) \in \Sigma_{out}\};$$

thus $\mathcal{R}(v) = \varphi_{t_{\mathcal{R}}(v)}(v)$, for all $v \in \Sigma_{in} \setminus Z^s$.

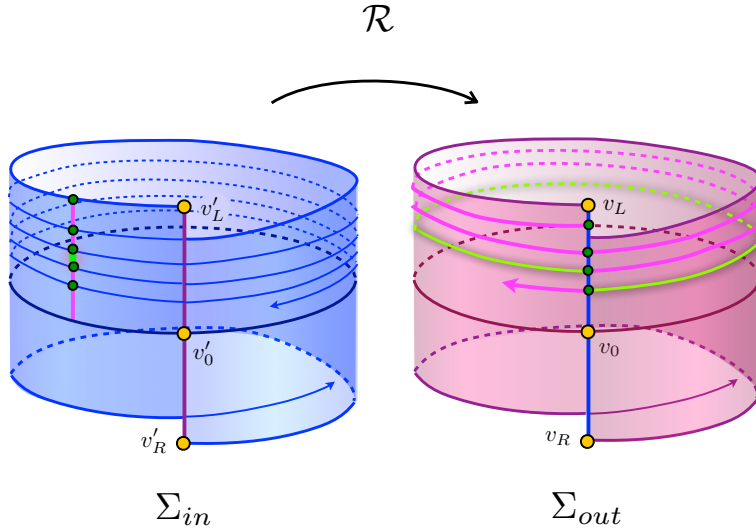


FIGURE 4. The action of \mathcal{R} on fundamental intervals.

Let $v \in Z^s$ and let $I \subset (\pi^u)^{-1}(v)$ be a closed interval. We say that I is a *fundamental interval* if the endpoints of I lie on the same leaf of the $\widehat{\mathcal{W}}^s$ foliation and the interior of I contains no points on that leaf.

Lemma 5.3. *There exists $C_1 \geq 1$ such that for any fundamental interval I , the following holds:*

- (1) $\pi^s(\mathcal{R}(I)) = Z^u$, and the restriction of $\pi^s \circ \mathcal{R}$ to the interior of I is a $C^{1+\alpha}$ diffeomorphism, whose inverse has uniformly bounded distortion.
- (2) For any $w \in I$, we have $\|\mathcal{R}'(w)\| \asymp |I|^{-1}$, where $\mathcal{R}'(w)$ denotes the derivative of the restriction of \mathcal{R} to I .

Proof. Property (1) follows from the construction of fundamental intervals and the fact that the foliation \mathcal{W}_φ^{cs} is uniformly $C^{1+\alpha}$. Property (2) follows from Corollary 4.16. \diamond

Lemma 5.4. *There exist $C_2 \geq 1$ and $\alpha > 0$ such that if I is a fundamental interval, and $b(I) = \inf_{v \in I} |b(v)|$, then*

$$|I| \leq C_2 |b(I)|^{1+\alpha}.$$

Proof. For a point $v \in I$, let t_1 be the return time of v for φ_t to Σ_{out} . Note that $\delta(\varphi_{t_1}(v)) = \delta(v)$ and $b(\varphi_{t_1}(v)) \approx b(v)$ (by Proposition 3.2). Let $t_0 \in (0, t_1)$ be the point where $a(\varphi_{t_0}(v)) = 0$.

We write $a(t), b(t), c(t)$ for $a(\varphi_t(v)), b(\varphi_t(v)), c(\varphi_t(v))$, respectively. Lemma 3.1 implies that $c = \frac{1}{|a|} \frac{b'}{b} \geq \frac{b'}{b}$, and so

$$\exp \left(\int_0^{t_0} c \right) \geq \exp \left(\int_0^{t_0} \frac{b'}{b} \right) = \exp(-\ln(b(0))) = |b(0)|^{-1}.$$

Thus, since $c = r/\delta + O(1)$ by Proposition 2.1, we have

$$\exp \left(\int_0^{t_1} \frac{r}{\delta} \right) \geq C^{-1} |b(0)|^{-2},$$

for some $C \geq 1$, and

$$\|\mathcal{R}'(w)\| \asymp \|D\varphi_{t_1} E^u\| \geq \exp \left(\int_0^{t_1} \frac{r-1-\epsilon}{\delta} \right) \geq C^{-1} |b(0)|^{-2(r-1-\epsilon)/r}.$$

Then, since distortion is bounded on small intervals by Corollary 4.16, we have

$$|I| \asymp \|\mathcal{R}'(w)\|^{-1} \leq C_2 |b(0)|^{2(r-1-\epsilon)/r} \leq C_2 |b(I)|^{1+\alpha},$$

for some $\alpha > 0$, since $r > 2$. \diamond

As a corollary, we obtain:

Lemma 5.5. *There exist $C \geq 1$ and $\alpha \in (0, 1)$ such that the following holds. If B is a \widehat{W}^u -interval in Σ_{in} with $Z^s \cap B \neq \emptyset$, and $I \subset B$ is a fundamental interval, then $|I| \leq C|B|^{1+\alpha}$. In particular, if $|B|$ is sufficiently small then $|I| \leq |B|/4$.*

5.2. Building a Young tower.

Lemma 5.6. *Fix a compact set $K \subset T^1S$, $\eta > 0$ and $\sigma > 0$. There exists $U_0 > 0$ such that for any $v \in K$, there exists $w \in \mathcal{W}^u(v, \sigma)$ and $t \in (0, U_0]$ such that $\varphi_t(w) \in \mathcal{T}_{in}(\delta_2, \eta)$.*

Proof. This is a consequence of ergodicity (indeed transitivity) of φ_t , compactness of K , and the Anosov condition on ψ_t .

Let η, σ and K be given. Let $\sigma_1 < \eta/8$ be small enough such that for all $v_1, v_2 \in K$, if $d(v_1, v_2) < \sigma_1$, then $\mathcal{W}^s(v_1, \sigma) \cap \mathcal{W}^{cu}(v_2, \sigma) \neq \emptyset$ (because ψ_t is Anosov, this holds for the restriction of the \star metric to K , which is then comparable to the original metric, since K is compact).

Since φ_t is ergodic (by Corollary 4.13) there exists $u \in T^1S$ whose backward orbit is dense in T^1S and such that

$$\varphi_{[-\sigma_1/2, \sigma_1/2]}(u) \cap \mathcal{T}_{in}(\delta_2, \eta/2) \neq \emptyset.$$

Cover the compact set K with a finite collection of $\sigma_1/2$ -balls. Fix $s_0 > 0$ such that $\varphi_{-s_0}(\mathcal{W}^s(u, \sigma_1/4))$ has length $> \sigma$. If s_1 is sufficiently large, then $\varphi_{[-s_1, -s_0]}(u)$ meets all of the $\sigma_1/2$ -balls, and thus meets every σ -center unstable manifold. This implies the conclusion, with $U_0 = s_1 + \sigma_1$. \diamond

We now describe the procedure for partitioning $\Delta_0 = Z^u \setminus \{v_0\}$ into a full measure set of subintervals $\{\Delta_j : j \geq 1\}$ mapping onto Δ_0 under $\pi^{cs} \circ \varphi_R$, where $R: \bigcup_j \Delta_j \rightarrow \mathbb{R}$.

We assume δ_2 and η_0 are very small, so that by Corollary 4.16 distortion is at most $4/3$ on unstable intervals of length $\leq 2\eta_0$:

$$w \in \mathcal{W}^u(v, 2\eta_0) \implies \frac{\|D_w^u \varphi_{-t}\|}{\|D_v^u \varphi_{-t}\|} \in \left[\frac{3}{4}, \frac{4}{3}\right], \forall t > 0.$$

Denote by $\ell > 0$ the circumference of Z^u , which is less than 1 if δ_2 is small enough, and without loss of generality assume $\eta_0 < \ell/2$. Let $t_{\mathcal{R}}: \Sigma_{in} \setminus Z^s \rightarrow \mathbb{R}$ be the return function defined by (41).

Fix $\Theta \subset T^1 S$ the thick part defined by

$$\Theta = \{v \in T^1 S : \bar{\delta}(v) \geq \delta_2\}.$$

Note that $\Sigma_{in} \cup \Sigma_{out} \subset \Theta$. Lemma 5.6 implies that there exists $U_0 > 0$ such that for any $v \in \Theta$, there exists $w \in \mathcal{W}^u(v, \eta_0)$ and $t \in (0, U_0]$ such that $\varphi_t(w) \in \mathcal{T}_{in}(\delta_2, \eta_0)$. Let s_0 be the maximum time needed for a piece of unstable interval to double in length under φ_t . Let $U = U_0 + s_0 + 2\delta_2$. Denote by Z^{cs} the singular set consisting of all vectors $v \in T^1 S$ with $\bar{\delta}(v) \leq \delta_2$ and such that $\varphi_t(v)$ hits the cusp in time $t \leq \delta_2$; that is:

$$Z^{cs} := \varphi_{[0, \delta_2]}(Z^s).$$

We begin by chopping Z^u into a collection \mathcal{G}_0 of intervals of length in $[\eta_0, 2\eta_0]$. We say that an open interval $G \subset Z^u$ is *active gap interval at time $t \geq 0$* if $\varphi_{[0, t]}(G) \cap Z^{cs} = \emptyset$ and $|\varphi_t(G)| \in [\eta_0, 2\eta_0]$. Thus \mathcal{G}_0 consists of active gap intervals at time 0.

Recall from Corollary 3.4 that for all $v \in \Sigma_{in} \setminus Z^s$, $\varphi_{2\delta_2}(v) \in \Theta$. This implies that if $G \subset Z^u$ is *any* piece of unstable manifold of length less than η_0 such that:

- $\varphi_{t_0}(G) \cap \Sigma_{in} \neq \emptyset$, and
- $\pi^{cs}(\varphi_{t_0}(G))$ contains a fundamental interval,

then there exists $t_1 \in (0, 2\delta_2)$ such that G is an active gap interval at time $t_0 + t_1$.

We now describe an algorithm for evolving an active gap interval to produce new active gap intervals and other intervals called border intervals.

Let $G \subset Z^u$ be an active gap interval at some time $t_0 \geq 0$. We then flow G forward until the first $t > t_0$ when one of two things happens:

- (a) $|\varphi_t(G)| = 2\eta_0$, or
- (b) $\varphi_t(G) \cap Z^s \neq \emptyset$.

Either (a) or (b) will occur within time s_0 .

If (a) happens first, we chop G into two new gap intervals, G_1 and G_2 , so that $|\varphi_t(G_1)| = |\varphi_t(G_2)| = \eta_0$. We say that G_1 and G_2 are *born* and become *active* at time t and write $t_b(G_1) = t_a(G_1) = t_a(G_2) = t_b(G_2) = t$. Note that $G = G_1 \cup \{v_0\} \cup G_2$, where v_0 is the point where the interval G is cut. Since distortion is bounded by $4/3$ on

intervals of length $\leq 2\eta_0$, we have:

$$(42) \quad \frac{|G_1|}{|G_2|} \in \left[\frac{3}{4}, \frac{4}{3}\right], \quad \text{and} \quad \frac{|G_i|}{|G|} \in \left[\frac{3}{8}, \frac{2}{3}\right], i = 1, 2.$$

We say that G is *inactive* in the time interval $[t, \infty)$, and that G_1, G_2 are inactive in the time interval $[0, t)$.

If (b) happens first, then G gives birth to two *border intervals* B_1 and B_2 and two gap intervals G_1, G_2 as follows. Let $v \in G$ be the unique point satisfying $\varphi_t(v) \in Z^s$. Let X_1 and X_2 be the components of $G \setminus \{v\}$ that lie to the left and right of v , respectively. For $i = 1, 2$, let $B_i \subset X_i$ be the largest interval satisfying:

- $\{v\} \cup B_i$ is a closed interval,
- $\pi^{cs}\varphi_t(B_i)$ is a countable union of fundamental intervals, and
- $\pi^{cs}\varphi_t(G_i)$ contains exactly one fundamental interval, where $G_i = X_i \setminus B_i$.

For $i = 1, 2$, we define the *birthday* of B_i and G_i to be $t_b(B_i) = t_b(G_i) = t$. Let $t_a(G_i)$ to be the first time such that $\varphi_{t_a(G_i)}$ is an active interval. Note that since $\varphi_t(G_i)$ meets Σ_{in} and $\pi^{cs}(\varphi_t(G_i))$ contains a fundamental interval, we have that $t_a(G_i) \in (t, t + 2\delta_2]$. Similarly, any fundamental interval in $\pi^{cs}\varphi_t(B_i)$ will return to Σ_{out} in time at most $2\delta_2$.

To summarize, in case b) we produce a decomposition of the original active gap interval G into disjoint subintervals

$$G = G_1 \cup B_1 \cup B_2 \cup G_2,$$

(up to a finite set of points) with the following properties:

- G_1 and G_2 are gap intervals that are born at time $t_b(G_1) = t_b(G_2) \in [t_0, t_0 + s_0]$, respectively. For $i = 1, 2$, there exists $t_a(G_i) \in [t_b(G_i), t_b(G_i) + 2\delta_2]$ such that G_i is active at time $t_a(G_i)$. We say that G_i is *inactive* in the time period $(0, t_b(G_i))$ and *dormant* during the period $[t_b(G_i), t_a(G_i))$.
- B_1 and B_2 are border intervals that are born at time $t_b(B_1) = t_b(B_2) \in [t_0, t_0 + s_0]$, respectively. For $i = 1, 2$, the set $\varphi_{t_b(B_i)}(B_i)$ is a countable union of fundamental intervals, and for any $v \in B_i$, we have $t_{\mathcal{R}}(\varphi_{t_b(B_i)}(v)) \in (0, 2\delta_2]$.
- Since each $\pi^{cs}\varphi_{t_b(G_i)}(G_i)$ contains exactly one fundamental interval (and no more), it has bounded length when it first returns to Θ : when this forward image is projected onto Z^u it covers at least once, but not more than twice. Thus, assuming that δ_1, η_0 etc. are small enough, we have that if $t_1 > t_b(G_i)$ is the smallest time such that $\varphi_{t_1}(G_i) \cap \Sigma_{out} \neq \emptyset$, then

$$(43) \quad |\varphi_{t_1}(G_i)| \in [\ell/2, 3\ell).$$

- Since each $\pi^{cs}\varphi_{t_b(G_i)}(G_i)$ is contained in two fundamental intervals, Lemma 5.5 implies that $|\varphi_{t_b(G_1)}(G_1 \cup G_2)| \leq \frac{1}{2}|\varphi_{t_b(G_1)}(G)|$; since distortion is bounded by $4/3$ on intervals of length $\leq 2\eta_0$, we have:

$$(44) \quad |G_1 \cup G_2| \leq \frac{2}{3}|G|.$$

Starting with the intervals in \mathcal{G}_0 and applying the algorithm to all active gap intervals, we obtain for any time $t \geq 0$, three disjoint collections of disjoint intervals $\mathcal{A}_t, \mathcal{B}_t$ and \mathcal{D}_t , the active, border and dormant intervals. The set \mathcal{A}_t consists of the gap intervals that are active at time t , the set \mathcal{B}_t consists of border intervals B with birth time $t_b(B) \leq t$,

and \mathcal{D}_t are the gap intervals that are dormant at time t . Let $\mathcal{G}_t = \mathcal{A}_t \cup \mathcal{D}_t$ be the collection of all gap intervals active or dormant at time t .

Observe that for any $t > 0$, we have

$$Z^u = \bigcup \mathcal{B}_t \cup \bigcup \mathcal{G}_t \cup \bigcup \mathcal{V}_t,$$

where \mathcal{V}_t is a finite collection of points. (Note that at $t = 0$, we have $\mathcal{G}_0 = \mathcal{A}_0$, $\mathcal{B}_0 = \mathcal{D}_0 = \emptyset$, and $\mathcal{V}_0 = \{v_0\}$).

Lemma 5.7. *There exists $\lambda_0 \in (0, 1)$ such that for any $k \geq 0$:*

$$\left| Z^u \setminus \bigcup \mathcal{B}_{kU} \right| = \left| \bigcup \mathcal{G}_{kU} \right| \leq \lambda_0^k.$$

Proof. Let $G \in \mathcal{G}_{kU}$. Then G is either active or dormant at time kU . Since an interval cannot be active for more than time $s_0 \leq U$ and cannot be dormant for more than time $2\delta_2 \leq U$, it follows that G is inactive at time $(k-1)U$. It follows that G has a unique ancestor in $\mathcal{G}_{(k-1)U}$; that is, there exists $G' \in \mathcal{G}_{(k-1)U}$ such that $G \subset G'$ and G' gives birth in the time interval $[(k-1)U, kU]$.

Now suppose $G' \in \mathcal{G}_{(k-1)U}$. Then G' will become active within time $2\delta_2$ and some point $v \in G'$ will intersect Z^s within time $2\delta_2 + U_0 \leq U$. Thus during the time period $[(k-1)U, kU]$, the interval G' will divide finitely many times, and at least one active piece will intersect Z^s .

The number of times this division can occur is uniformly bounded. As the gap evolves in the time interval $[(k-1)U, kU]$, it gives birth to new gaps according to rule (a) or (b) above. The number of times that case (a) can apply between two occurrences of case (b) is bounded: if a gap G'' is produced by rule (b), then by (43), we have $|\varphi_{t_1}(G'')| \in [\ell/2, 3\ell]$, where $t_1 \geq t_b(G'')$ is the first time G'' returns to Σ_{out} . In Θ , the derivative $D\varphi_t$ is bounded above, and so any active interval meeting Θ can divide a bounded number of times before some descendent meets Z^s (which happens within time U_0). Thus the number of times (a) can apply within two occurrences of (b) is uniformly bounded.

We conclude that $G' = G_1 \cup \dots \cup G_n \cup B_1 \dots \cup B_{2m}$, with $n \geq 2m \geq 2$, where $G_1, \dots, G_n \in \mathcal{G}_{kU}$, and $B_1, \dots, B_{2m} \in \mathcal{B}_{kU}$. Moreover, there exists $N > 0$, independent of k, G' such that $n \leq Nm$. Combined with (42) and (44), this implies that there exists $\lambda_0 \in (0, 1)$ such that

$$|G_1 \cup \dots \cup G_n| \leq \lambda_0 |G'|.$$

Thus $|\mathcal{G}_{kU}| \leq \lambda_0 |\mathcal{G}_{(k-1)U}|$; since $|\mathcal{G}_0| < 1$, we obtain the conclusion. \diamond

Let $\mathcal{B}_\infty = \bigcup_{t>0} \mathcal{B}_t$. Then \mathcal{B}_∞ is a collection of disjoint intervals with $|Z^u \setminus \bigcup \mathcal{B}_\infty| = 0$.

Proof of Theorem 5.1. We create a countable collection \mathcal{I} of intervals Δ_j as follows: we decompose each $B \in \mathcal{B}_\infty$ into a countable union $B = \bigsqcup_{j \geq 1} \Delta_{B,j}$ such that for each j , $\pi^{cs} \varphi_{t_b(B)}(\Delta_{B,j})$ is a fundamental interval. Then we set

$$\mathcal{I} = \{\Delta_{B,j} : B \in \mathcal{B}_\infty, j \geq 1\}.$$

Note that $|Z^u \setminus \bigcup \mathcal{I}| = |Z^u \setminus \bigcup \mathcal{B}_\infty| = 0$, and so conclusion (1) of Theorem 5.1 holds.

We extend the definition of t_b to intervals in \mathcal{I} in the natural way: if $\Delta_j \subset B \in \mathcal{B}_\infty$, we set $t_b(\Delta_j) = t_b(B)$. For $v \in \Delta_j \subset \mathcal{I}$, let

$$R_0(v) = t_b(\Delta_j) + t_{\mathcal{R}}(\varphi_{t_b(\Delta_j)}(v)).$$

Then $R_0(v)$ is the minimal time $> t_b(\Delta_j)$ such that $\varphi_{R_0(v)}(v) \in \Sigma_{out}$.

Lemma 5.8. *There exist $\lambda \in (0, 1)$ and $C \geq 1$ such that the function $R_0: \bigcup \mathcal{I} \rightarrow \mathbb{R}_{>0}$ satisfies*

$$\left| \left\{ v \in \bigcup \mathcal{I} : R_0(v) \geq k \right\} \right| \leq C\lambda^k,$$

for each $k \geq 0$.

Proof. Since $t_{\mathcal{R}}$ is bounded it suffices to find $\lambda_1 \in (0, 1)$ such that

$$\left| \bigcup \{ B \in \mathcal{B}_{\infty} : t_b(B) \geq k \} \right| \leq C\lambda_1^k.$$

But this follows immediately from the construction with $\lambda_1 = \lambda_0$ appearing in Lemma 5.7. \diamond

We now define the return time function $R: \bigcup \mathcal{I} \rightarrow \mathbb{R}_{>0}$. Recall the projection $\pi^s: \Sigma_{out} \rightarrow Z^u$ along the leaves of $\widehat{\mathcal{W}}^s$. The fibers of π^s are local $\widehat{\mathcal{W}}^s$ manifolds. Over each point $v \in \Sigma_{out}$ there lies a unique point $w(v) \in \mathcal{W}_{loc}^s(\pi^s(v))$ such that $w(v) = \varphi_{r(v)}(v)$, for some small value of $r(v)$. Let $\bar{\Sigma}_{out} = w(\Sigma_{out})$.

Lemma 5.9. *The function $r: \Sigma_{out} \rightarrow \mathbb{R}$ is uniformly $C^{1+\alpha}$, and $\bar{\Sigma}_{out}$ is a $C^{1+\alpha}$ manifold. The map $w: \Sigma_{out} \rightarrow \bar{\Sigma}_{out}$ is a $C^{1+\alpha}$ diffeomorphism. The manifold $\bar{\Sigma}_{out}$ is $C^{1+\alpha}$ foliated by local \mathcal{W}^s -leaves:*

$$\bar{\Sigma}_{out} \subset \bigcup_{v' \in \Delta_0} \mathcal{W}_{loc}^s(v'),$$

and the projection $\bar{\pi}^s: \bar{\Sigma}_0 \rightarrow Z^u$ along these local leaves is a $C^{1+\alpha}$ submersion.

Proof. This follows from the fact that the foliation \mathcal{W}^s is uniformly $C^{1+\alpha}$. \diamond

We define $R: \bigcup \mathcal{I} \rightarrow \mathbb{R}$ by $R(v) = R_0(v) + r(\varphi_{R_0(v)}(v))$; it has the property that $\varphi_{R(v)}(v) \in \bar{\Sigma}_{out}$. Lemma 5.9 implies that for each $v \in \bigcup \mathcal{I}$, there exists a unique $v' \in \Delta_0$ – namely, $v' = \bar{\pi}^s \varphi_{R(v)}(v)$ – such that $\varphi_{R(v)}(v) \in \mathcal{W}_{loc}^s(v')$, giving conclusion (2).

For $\Delta_j \in \mathcal{I}$, we define $h_j: \Delta_0 \rightarrow \Delta_j$ to be the inverse of the map $F_j = \pi^{cs} \circ \varphi_{R(\cdot)} = \bar{\pi}^s \circ \varphi_{R(\cdot)}: \Delta_j \rightarrow \Delta_0$. This is well-defined, because

$$\bar{\pi}^s(\varphi_{R(\cdot)}(\Delta_j)) = \pi^s(\mathcal{R}(\pi^{cs} \circ \varphi_{t_b(\Delta_j)}(\Delta_j))) = \Delta_0,$$

since $\pi^{cs} \varphi_{t_b(\Delta_j)}(\Delta_j)$ is a fundamental interval. Since π^s is a submersion, and the map $v \mapsto \mathcal{R}(\pi^{cs} \varphi_{t_b(B)})(v)$ is a diffeomorphism from Δ_j onto its image, the composition is a diffeomorphism from Δ_j to Δ_0 . Thus its inverse $h_j: \Delta_0 \rightarrow \Delta_j$ is a diffeomorphism. This establishes conclusion (3).

Note that $|h'_j(v)| \sim \|D_{h_j(v)}^u \varphi_{R_0(h_j(v))}\|^{-1}$, and so conclusion (4) holds. Conclusion (5) follows from the facts that φ_t has bounded distortion and the map π^{cs} is uniformly $C^{1+\alpha}$. Indeed note that the map $F_j = h_j^{-1}$ can also be expressed in the following way. We fix some point $\hat{v} \in \Delta_j$ and consider the image $\varphi_{R(\hat{v})}(\Delta_j)$ under the constant time flow $\varphi_{R(\hat{v})}$. This is a piece of unstable manifold that meets $\bar{\Sigma}_{out}$. The map $F_j = h_j^{-1}$ is just the composition $F_j = \pi^{cs} \circ \varphi_{R(\hat{v})}$ of this flow with the center-stable projection $\pi^{cs}: U \rightarrow Z^u$, defined in the beginning of the section. This latter projection is a uniformly $C^{1+\alpha}$ submersion and a local diffeomorphism when restricted to local unstable manifolds, since the foliation \mathcal{W}^{cs} is uniformly $C^{1+\alpha}$. Thus h_j is uniformly $C^{1+\alpha}$.

Let's examine the map $R \circ h_j$. Again fix a point $\hat{v} \in \Delta_j$, and consider the image $\varphi_{R_0(\hat{v})}(\Delta_j)$, which is a piece of unstable manifold meeting Σ_{out} at the point $\varphi_{R_0(\hat{v})}(\hat{v})$. It follows that there is a uniformly bounded C^2 function $\hat{r}: \varphi_{R_0(\hat{v})}(\Delta_j) \rightarrow \mathbb{R}$ such that $\varphi_{\hat{r}(\cdot)}$ sends the piece of unstable manifold $\varphi_{R_0(\hat{v})}(\Delta_j)$ to $\widehat{\mathcal{W}}_{loc}^u(\varphi_{R_0(\hat{v})}(\hat{v})) \subset \Sigma_{out}$. Then $R_0(v) = R_0(\hat{v}) + \hat{r}(\varphi_{R_0(\hat{v})}(v))$, and so $R(v) = R_0(\hat{v}) + \hat{r}(\varphi_{R_0(\hat{v})}(v)) + r(\varphi_{R_0(v)}(v))$. Thus

$$\begin{aligned} |R'(v)| &\leq |\hat{r}'(\varphi_{R_0(\hat{v})}(v))| \|D_v^u \varphi_{R_0(\hat{v})}\| \\ &\quad + |r'(\varphi_{R_0(v)}(v))| (\|D_v^u \varphi_{R_0(v)}\| + \|\dot{\varphi}(\varphi_{R_0(v)}(v))\| |\hat{r}'(\varphi_{R_0(\hat{v})}(v))| \|D_v^u \varphi_{R_0(\hat{v})}\|) \\ &= |\hat{r}'(\varphi_{R_0(\hat{v})}(v))| \|D_v^u \varphi_{R_0(\hat{v})}\| (1 + |r'(\varphi_{R_0(v)}(v))|) + |r'(\varphi_{R_0(v)}(v))| \|D_v^u \varphi_{R_0(v)}\|. \end{aligned}$$

The derivatives r' and \hat{r}' are uniformly bounded. Since $|h'_j(v)| \asymp \|D_{h_j(v)}^u \varphi_{R_0(h_j(v))}\|^{-1}$, we obtain that there exists a uniform constant $C \geq 1$ such that $|(R \circ h_j)'| \leq C$, for all j . This gives conclusion (6).

Since the function r is bounded, Lemma 5.8 implies that for each $k > 0$, we have

$$\left| \left\{ v \in \bigcup \mathcal{I} : R(v) \geq k \right\} \right| \leq C \lambda^k;$$

since $|h'_j| \asymp |\Delta_j|$, this gives conclusion (7) of Theorem 5.1, where $\epsilon > 0$ is chosen so that $\lambda \exp(C\epsilon) < 1$.

Finally we verify that the UNI Condition in conclusion (8) holds. This is a direct consequence of the fact that φ_t preserves a contact 1-form ω , which implies that the foliations \mathcal{W}^s and \mathcal{W}^u are not jointly integrable. The details are carried out in Lemma 12 of [1] (in the Axiom A context) and Lemma 4.2 and Corollary 4.3 of [4] (close to the current context). \diamond

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